Math 7770 Spring 2011
Homework 2
Due: January 31, 2011

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books outside of our textbook or other people, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

Warmup

- §1.3 #1. With $a = \prod_p p^{\alpha(p)}$ and $b = \prod_p p^{\beta(p)}$ (as in equation (1.6) in the book) what conditions on the exponents must be satisfied if $(a, b) = 1$?
- §1.3 #10. Prove that any positive integer of the form $3k+2$ has a prime factor of the same form; similarly for each of the forms $4k + 3$ and $6k + 5$.
- §1.3 #16. Find a positive integer n such that $n/2$ is a square, $n/3$ is a cube, and $n/5$ is a fifth power.
- §1.3 #22. See book for this one. There are a whole series of True/False questions about primes and divisibility.
- §2.1 # 5. Write a single congruence that is equivalent to the pair of congruences $x \equiv 1 \bmod 4$ and $x \equiv 2 \bmod 3$
- §2.1 # 6. Prove that if p is a prime and $a^2 \equiv b^2 \bmod p$ then $p \mid (a + b)$ or $p \mid (a - b)$.
- §2.1 #10. Evaluate $\phi(m)$ for $m = 1, 2, 3, \cdots , 12$.

Problems

1. §1.3 #4. Prove that an integer is divisible by 3 if and only if the sum of its digits is divisible by 3. Prove that an integer is divisible by 9 if and only if the sum of its digits is divisible by 9.
2. §1.3 #19. Let a and b be positive integers such that $(a, b) = 1$ and ab is a perfect square. Prove that a and b are perfect squares. Prove that the result generalizes to kth powers.
3. §1.3 #26. Prove that there are infinitely many primes of the form $4n + 3$ or $6n + 5$. (Hint: Use a variation of the proof that there are an infinite number of primes and see §1.3 #10 above.)
4. §2.1 #40. For m odd, prove that the sum of the elements of any complete residue system modulo m is congruent to zero modulo m; prove the analogous result for any reduced residue system for $m > 2$.

Challenge

I. §1.3 #51. Show that 24 is the largest integer divisible by all integers less than its square root. (Hint: Consider the highest power of 2, of 3, of 5, of 7 less than the square root.)