Math 7770 Spring 2011
Homework 11
Due: April 18, 2011

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books outside of our textbook or other people, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

Warmup

- §9.7 # 3. Prove that $11 + 2\sqrt{6}$ is irreducible in $\mathbb{Q}(\sqrt{6})$. (Book says prime.)
- Show that $1 - i$ is irreducible in $\mathbb{Z}[i]$ and that $2 = u(1 - i)^2$ for some unit u.
- Let $R = \mathbb{Z}[\sqrt{-n}]$ where n is a square-free integer greater than 3. Prove that 2, $\sqrt{-n}$, and $1+\sqrt{-n}$ are irreducibles in R.
- Let $K = \mathbb{Q}(\sqrt{10})$. Show that $4 - \sqrt{10}$ and $98 + 31\sqrt{10}$ are associates in $\mathcal{O}_K$. Hint: $3 + \sqrt{10}$ is a unit.

Problems

1. §9.7 # 4. Prove that 3 is irreducible in $\mathbb{Q}(i)$ but not irreducible in $\mathbb{Q}(\sqrt{6})$. (Book says prime.)
2. Find all solutions to the equation $y^2 + 1 = x^3$ in rational integers.

   Here are some steps to do this:
   - Factor the left hand side in $\mathbb{Q}(\sqrt{-1})$.
   - Show that y is even.
   - Show that the factors you obtained in (a) have no common factors except units.
   - Conclude that each of the factors in (a) is, up to units, the cube of an integer.
   - Equate coefficients and find solution(s).
3. Prove that $\mathbb{Q}(\sqrt{10})$ does not have unique factorization.
4. Notice that $(2 + i)(2 - i) = 5 = (1 + 2i)(1 - 2i)$. How is this consistent with unique factorization? Justify your answer.