Math 704 Summer 2008
Homework 6
Due: July 28, 2008

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books or other people heavily, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

1. Section 5.2 # 8. Let A be a finite abelian group (written multiplicatively) and let p be a prime. Define

$$A^p = \{a^p \mid a \in A\} \text{ and } A_p = \{x \mid x^p = 1\}.$$

(a) Prove that $A/A^p \cong A_p$.
(b) Prove that the number of subgroups of A of order p equals the number of subgroups of A of index p. (Hints in the book.)

2. Section 5.4 # 6. Exhibit a representative of each cycle type of A_5 as a commutator in S_5.

3. Section 5.4 # 7. Prove that if p is a prime and P is a non-abelian group of order p^3 then $P' = Z(P)$.

4. Section 5.4 # 15. If A and B are normal subgroups of G such that G/A and G/B are both abelian, prove that $G/(A \cap B)$ is abelian.

5. Let G be a group and let H be a cyclic normal subgroup of G. Prove that the commutator subgroup G' is a subgroup of the centralizer $C_G(H)$. (Hint: Note that $\mathrm{Aut}(H)$ is an abelian group).

6. Let F be a field (see section 1.4 in the book for brief definition of field) and show that the matrix group G is a solvable group where

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F, ac \neq 0 \right\}.$$

7. Let G be a finite nilpotent group and let $x, y \in G$ be fixed elements. Assume that for any normal subgroup N of G, $x \in N$ whenever $[x, y] \in N$. Show that $x = 1_G$.

8. Section 6.1 # 7. (First part only) Prove that subgroups and quotient groups of nilpotent groups are nilpotent (do not assume finiteness).