Math 704 Summer 2008
Homework 2
Due: June 23, 2008

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books or other people heavily, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

1. Section 2.1 # 6. Let G be an abelian group. Prove that $\{g \in G \mid |g| < \infty\}$ is a subgroup of G (called the **torsion subgroup** of G). Give an explicit example where this set is not a subgroup when G is non-abelian.

2. Section 2.1 # 9. Let $G = \mathrm{GL}_n(F)$ for a field F. Define the **special linear group**

$$\mathrm{SL}_n(F) = \{A \in \mathrm{GL}_n(F) \mid \det(A) = 1\}.$$

   Prove that $\mathrm{SL}_n(F) \leq \mathrm{GL}_n(F)$.

3. Section 2.1 # 11. If A and B are groups, prove the following sets are subgroups of the direct product $A \times B$.
   (a) $\{(a, 1) \mid a \in A\}$
   (b) $\{(1, b) \mid b \in B\}$
   (c) $\{(a, a) \mid a \in A\}$ where we assume $B = A$.

4. Section 2.2 # 2. Prove that $C_G(Z(G)) = G$ and deduce that $N_G(Z(G)) = G$.

5. Section 2.2 # 5. In each case below, show that for the specified group G and subgroup A of G, $C_G(A) = A$ and $N_G(A) = G$.
   (a) $G = S_3$ and $A = \{1, (1\ 2\ 3), (1\ 3\ 2)\}$
   (b) $G = D_8$ and $A = \{1, s, r^2, sr^2\}$
   (c) $G = D_{10}$ and $A = \langle r \rangle$

6. Section 2.3 # 3. Find all generators for $\mathbb{Z}/48\mathbb{Z}$.

7. Section 2.3 # 19. Show that if H is any group and h is an element of H, then there is a unique homomorphism from $\mathbb{Z}$ to H such that $1 \to h$.

8. (a) Convince yourself of Section 2.4 #8 ($S_4 = \langle (1\ 2\ 3\ 4), (1\ 2\ 4\ 3) \rangle$) and Section 2.4 #9 $\left(\mathrm{SL}_2(\mathbb{F}_3) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle\right)$. (You do not need to write this up).
   (b) Section 2.4 #11. Show that $\mathrm{SL}_2(\mathbb{F}_3)$ and S_4 are two nonisomorphic groups of order 24.

9. Section 2.4 # 15. Exhibit a proper subgroup of $\mathbb{Q}$ which is not cyclic.