Math 321: Foundations of Abstract Algebra

Homework 5 : Due February 28

The symbols $\oplus$ and $\odot$ specifically refer to the operations on the equivalence classes formed from the equivalence relation $a \sim_n b$ if $n \mid a - b$. And remember that elements of Z/nZ and $U(n)$ are *equivalence classes* so should be written with bars over them. E.g. $\mathbb{Z}/6\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ and $U(6) = \{\bar{1}, \bar{5}\}$.

1. #9.14
2. (8 points) (a) #4.4 (Don't forget to write $\bar{3}$, etc. for elements in $\mathbb{Z}/30\mathbb{Z}$.)

   (b) What is the order of $\overline{17}$ in $U(30)$?

   (c) Find a cyclic subgroup of order 4 in $U(40)$ and find a noncylic subgroup of order 4 in $U(40)$. (Don't forget to explain why.)
3. Let n and a be positive integers. Prove that the equation $\overline{a} \odot \overline{x} = \overline{1}$ has a solution $\overline{x}$ if and only if $\gcd(a, n) = 1$.
4. # 9.5 (For Q_8, $-I$ in the book is what I call -1 and J is what I call i. I define Q_8 as generated by i, j, and k, where $i^2 = j^2 = k^2 = -1$ and $ij = k$, $jk = i$, and $ki = j$, and $(-1)^2 = 1$.)
5. Let $G = S_4$ and let $H = \{e, (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}$. Compute the right and left cosets decompositions determined by H. Explain how you got the cosets, and why you know you have all of them. Theory is your friend here. Be as efficient as you can be!
6. Let G be a group and let H and K be subgroups of G. Let $a \in G$. Show that the two sets $Ha \cap Ka$ and $(H \cap K)a$ are equal. Thus the right cosets of the subgroup $H \cap K$ are obtained by intersecting the corresponding right cosets of H and K individually.
7. # 10.5
8. # 10.7
9. Let G be a group of order p^k where p is a prime and k is a positive integer. Show that G must have a subgroup of order p.