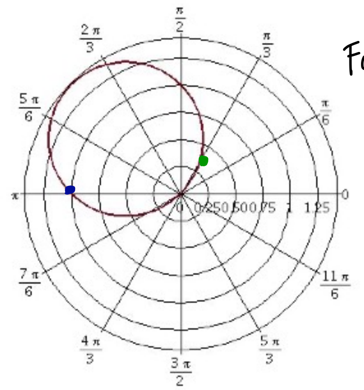
Final Exam Review Solutions

1. (a) Sketch the curve with the polar equation $r = \sin\theta - \cos\theta$.



For example, when $\theta = 0$, $r = \sin 0 - \cos 0 = 0 - 1 = -1$.
We note the point $(-1, 0)$ in blue.
Or when $\theta = \pi/3$, $r = \sin\frac{\pi}{3} - \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2} \approx 0.366$
We note the point $\left(\frac{\sqrt{3}-1}{2}, \pi/3\right)$ in green.

(b) How would you describe the line $y = \sqrt{3}x$ in polar coordinates?

We could let $x = r\cos\theta$ and $y = r\sin\theta$ to get $r\sin\theta = \sqrt{3}\, r\cos\theta$

so $\tan\theta = \sqrt{3}$ or $\boxed{\theta = \pi/3}$

(c) What's another way to describe the line in (b) in polar coordinates?

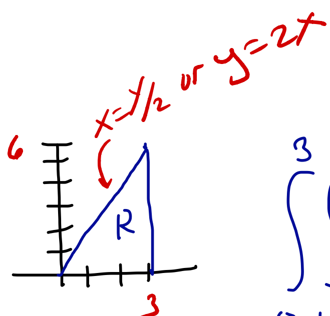
We could add or subtract π. So

$\theta = \pi/3 + \pi = \boxed{4\pi/3}$ or $\theta = \pi/3 - \pi = \boxed{-2\pi/3}$

2. Evaluate the integral $\displaystyle\int_0^6 \int_{y/2}^3 \frac{y}{x^3+1}\,dx\,dy.$

We can't integrate this the way it is

so we switch the order of integration.



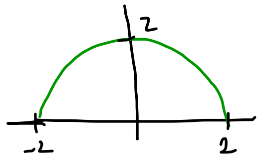
$$\int_0^3 \int_{y=0}^{y=2x} \frac{y}{x^3+1}\,dy\,dx = \int_0^3 \left.\frac{y^2}{2(x^3+1)}\right]_{y=0}^{y=2x} dx = \int_0^3 \frac{4x^2}{2(x^3+1)}\,dx$$

$u = x^3+1$
$du = 3x^2\,dx$

$$= \frac{2}{3}\ln|x^3+1|\Big]_0^3 = \boxed{\frac{2}{3}\ln 28} - \frac{2}{3}\ln \cancel{1}$$
0

3. Evaluate the integral $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x.$

We need to convert to polar.



$$= \int_0^{\pi}\int_0^{2} r\cdot r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^3}{3}\right]_0^2 d\theta = \int_0^{\pi}\frac{8}{3}\,d\theta =$$

$$\frac{8}{3}\theta\Big]_0^{\pi} = \boxed{\frac{8}{3}\pi}$$

4. Calculate the following integrals.

(a) $$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\int_{0}^{\sqrt{2}} \frac{y}{1+x^2}\,\mathrm{d}y\,\mathrm{d}x = \int_{1/\sqrt{3}}^{\sqrt{3}}\left[\frac{y^2}{2(1+x^2)}\right]_0^{\sqrt{2}} dx = \int_{1/\sqrt{3}}^{\sqrt{3}}\frac{2}{2(1+x^2)} - 0\,dx$$

$$= \int_{1/\sqrt{3}}^{\sqrt{3}}\frac{1}{1+x^2}\,dx = \tan^{-1}x\Big]_{1/\sqrt{3}}^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1}1/\sqrt{3}$$

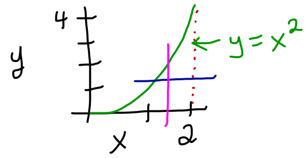
$$= \pi/3 - \pi/6 = \boxed{\pi/6}$$

(b) $$\int_{2}^{5}\int_{1}^{4} \frac{x}{y}+\frac{y}{x}\,\mathrm{d}y\,\mathrm{d}x = \int_2^5 x\ln y + \frac{y^2}{2x}\Big]_{y=1}^{y=4} dx = \int_2^5 x\ln 4 + \frac{8}{x} - x\underbrace{\cancel{\ln 1}}_{0} - \frac{1}{2x}\,dx$$

$$= \int_2^5 x\ln 4 + \frac{15}{2x}\,dx = \frac{\ln 4}{2}x^2 + \frac{15}{2}\ln x\Big]_2^5$$

$$= \boxed{\frac{\ln 4}{2}\cdot 25 + \frac{15}{2}\ln 5 - \frac{\ln 4}{2}\cdot 4 - \frac{15}{2}\ln 2}$$

(c) $\displaystyle\iint_R x\cos y\,\mathrm{d}A$ where R is the region bounded by $y = 0$, $y = x^2$ and $x = 2$.



$$\int_0^4\int_{x=\sqrt{y}}^{x=2} x\cos y\,dx\,dy \quad \text{or} \quad \int_0^2\int_{y=0}^{y=x^2} x\cos y\,dy\,dx$$

We'll solve blue.

$$\int_0^4 \frac{x^2}{2}\cos y\Big]_{x=\sqrt{y}}^{x=2} dy = \int_0^4 \frac{4}{2}\cos y - \frac{y}{2}\cos y\,dy$$

parts

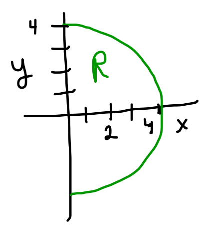
$u = y/2 \quad v = \sin y$

$du = 1/2\,dy \quad dv = \cos y\,dy$

$$= 2\sin y\Big]_0^4 - \left(\frac{y}{2}\sin y\Big]_0^4 - \int_0^4 \frac{1}{2}\sin y\,dy\right) = 2\sin y - \frac{y}{2}\sin y - \frac{1}{2}\cos y\Big]_0^4$$

$$= \cancel{2\sin 4} - \cancel{2\sin 4} - \frac{1}{2}\cos 4 - \left(0 - 0 - \frac{1}{2}\underbrace{\cancel{\cos 0}}_{1}\right) = \boxed{\frac{1}{2} - \frac{1}{2}\cos 4}$$

(d) $\displaystyle\iint_R e^{-x^2-y^2}\mathrm{d}A$ where R is the region bounded by the semicircle $x = \sqrt{16 - y^2}$ and the y-axis.



polar coordinates

$$\int_{-\pi/2}^{\pi/2}\int_0^4 e^{-(r\cos\theta)^2-(r\sin\theta)^2}\cdot r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^4 e^{-r^2(\overbrace{\cancel{\cos^2\theta+\sin^2\theta}}^{1})}\cdot r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^4 re^{-r^2}\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} -\frac{1}{2}e^{-r^2}\Big]_0^4 d\theta$$

$u = r^2$

$du = 2r\,dr$

$$= \int_{-\pi/2}^{\pi/2} -\frac{1}{2}e^{-16} + \frac{1}{2}\,d\theta = -\frac{1}{2}e^{-16}\theta + \frac{1}{2}\theta\Big]_{-\pi/2}^{\pi/2} = -\frac{\pi}{4}e^{-16} + \frac{\pi}{4} - \frac{\pi}{4}e^{-16} + \frac{\pi}{4}$$

$$= \boxed{\pi/2 - \pi/2\, e^{-16}}$$

5. Evaluate the following integrals

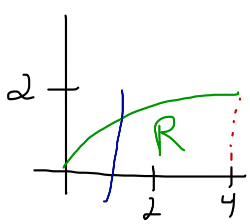
(a) $\int_{-1}^{1}\int_{2}^{4}\int_{0}^{2}\frac{x}{(y+z)^2}\,dx\,dy\,dz.$ $= \int_{-1}^{1}\int_{2}^{4}\frac{x^2}{2(y+z)^2}\Big]_0^2 dy\,dz = \int_{-1}^{1}\int_{2}^{4}\frac{2}{(y+z)^2}\,dy\,dz$

$$= \int_{-1}^{1}\frac{-2}{(y+z)}\Big]_{y=2}^{y=4} dz = \int_{-1}^{1}\frac{-2}{4+z}+\frac{2}{2+z}\,dz =$$

$$-2\ln|4+z|+2\ln|2+z|\Big]_{-1}^{1} = -2\ln5+2\ln3+2\ln3-2\overset{0}{\cancel{\ln 1}}$$

$$=\boxed{4\ln3-2\ln5}$$

(b) $\iiint_R 3xy\,dV$ where R lies under the plane $z=5+x+y$ and above the region in the xy -plane bounded by the curves $y=\sqrt{x}$, $y=0$ and $x=4$.



$$\int_{x=0}^{x=4}\int_{y=0}^{y=\sqrt{x}}\int_{z=0}^{z=5+x+y}3xy\,dz\,dy\,dx = \int_{0}^{4}\int_{y=0}^{y=\sqrt{x}}3xyz\Big]_0^{5+x+y} dy\,dx =$$

$$\int_{0}^{4}\int_{y=0}^{y=\sqrt{x}}3xy(5+x+y)\,dy\,dx = \int_0^4 \frac{15xy^2}{2}+\frac{3x^2y^2}{2}+\frac{\cancel{3}xy^3}{\cancel{3}}\Big]_{y=0}^{y=\sqrt{x}}dx$$

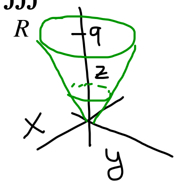
$$= \int_0^4 \frac{15}{2}x^2+\frac{3}{2}x^3+x^{5/2}\,dx = \frac{\overset{5}{\cancel{15}}}{2\cdot\cancel{3}}x^3+\frac{3}{2\cdot4}x^4+\frac{x^{7/2}}{7/2}\Big]_0^4$$

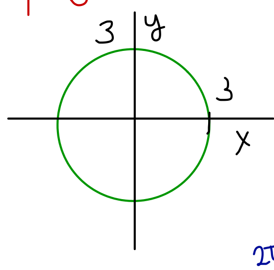
$$= \frac{5}{2}(4)^3+\frac{3}{8}(4)^4+\frac{2}{7}(4)^{7/2} = \boxed{160+96+\frac{256}{7}}$$

Cone is
$x^2+y^2=z$

(c) $\displaystyle\iiint_R z$ where R is the region between $x^2+y^2=z$ and $z=9$.

cylindrical coordinates
project onto xy





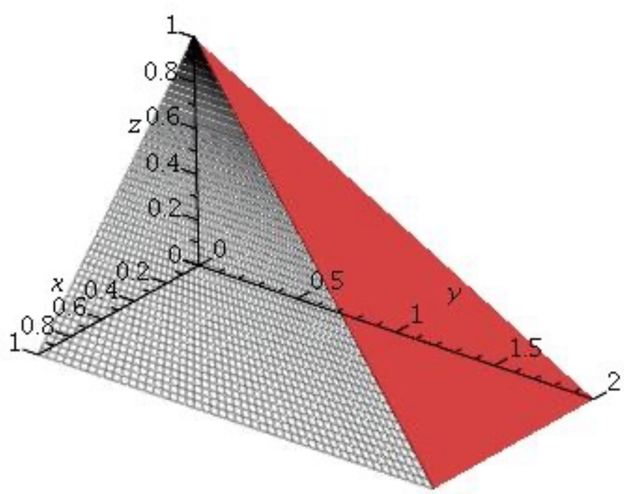
$r=\sqrt{x^2+y^2}$ so $r^2=x^2+y^2$

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=3}\int_{z=r^2}^{z=9} z\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{3}\left[\frac{z^2}{2}r\right]_{z=r^2}^{z=9} dr\,d\theta = \int_0^{2\pi}\int_0^{3}\frac{81}{2}r-\frac{r^5}{2}\,dr\,d\theta$$

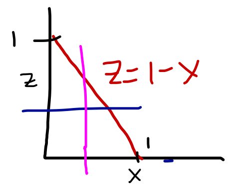
$$=\int_0^{2\pi}\left[\frac{81}{4}r^2-\frac{r^6}{12}\right]_0^3 d\theta = \int_0^{2\pi}\frac{3^6}{4}-\frac{3^{\cancel{6}}}{\cancel{12}}\;\frac{3^5}{4}\,d\theta = \int_0^{2\pi}\frac{3^5\cdot\cancel{2}}{\cancel{4}\,2}\,d\theta = \frac{243}{2}\theta\Big]_0^{2\pi}$$

$= \boxed{243\pi}$

6. Let R be the region in the first octant bounded by the planes $z=1-x$ and $y=2-2z$. (See picture below.) Express, **but do not evaluate** the triple integrals $\displaystyle\iiint_R f(x,y,z)\mathrm{d}V$ as an iterated integral in each of the six possible ways.



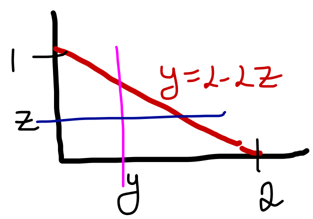
I. Project onto xz



$$\int_{z=0}^{z=1}\int_{x=0}^{x=1-z}\int_{y=0}^{y=2-2z} f(x,y,z)\,dy\,dx\,dz$$

$$\int_{x=0}^{x=1}\int_{z=0}^{z=1-x}\int_{y=0}^{y=2-2z} f(x,y,z)\,dy\,dz\,dx$$

II Project onto yz

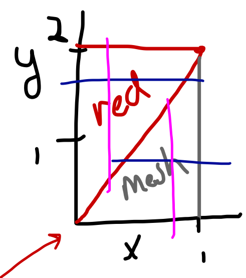


$$\int_{z=0}^{z=1}\int_{y=0}^{y=2-2z}\int_{x=0}^{x=1-z} f(x,y,z)\,dx\,dy\,dz$$

$$\int_{y=0}^{y=2}\int_{z=0}^{z=\frac{2-y}{2}}\int_{x=0}^{x=1-z} f(x,y,z)\,dx\,dz\,dy$$

III Project onto xy

We need to split this one up.



$$\int_{y=0}^{y=2}\int_{x=0}^{x=y/2}\int_{z=0}^{z=\frac{2-y}{2}} f(x,y,z)\,dz\,dx\,dy + \int_{y=0}^{y=2}\int_{x=y/2}^{x=1}\int_{z=0}^{z=1-x} f(x,y,z)\,dz\,dx\,dy$$

red

mesh

$$\int_{x=0}^{x=1}\int_{y=2x}^{y=2}\int_{z=0}^{z=\frac{2-y}{2}} f(x,y,z)\,dz\,dy\,dx + \int_{x=0}^{x=1}\int_{y=0}^{y=2x}\int_{z=0}^{z=1-x} f(x,y,z)\,dz\,dy\,dx$$

The divider is where $z=1-x$ and $z=\frac{2-y}{2}$ so

$$\cancel{1}-x=\frac{2-y}{2}=\cancel{1}-\frac{y}{2}$$

$$y=2x$$