Exam 3 Review Solutions

1. Let $f(x, y) = 4x - 3x^3 - 2xy^2$.

(a) Find the critical points of $f(x, y)$.

$f_x(x,y) = 4 - 9x^2 - 2y^2$

$f_y(x,y) = -4xy$

when $x = 0$, $4 - 2y^2 = 0$

$4 = 2y^2$

$y = \pm\sqrt{2}$

when $y=0$ $4 - 9x^2 = 0$

$4 = 9x^2$

$x = \pm 2/3$

So $\boxed{(0, \pm\sqrt{2}) \quad (\pm 2/3, 0)}$

(b) Are they local minima, local maxima, or saddle points? Why?

$f_{xx} = -18x \quad f_{yy} = -4x \quad f_{xy} = -4y$

at $(0, \sqrt{2})$ $D = \boxed{- \text{ so saddle point}}$

at $(0, -\sqrt{2})$ $D = \boxed{- \text{ so also saddle}}$

at $(2/3, 0)$ $D = +$ $f_{xx} = \boxed{- \text{ so max}}$

at $(-2/3, 0)$ $D = +$ $f_{xx} = \boxed{+ \text{ so min}}$

2. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = x^2 + 6x + 6y^2$ subject to the constraint $2x^2 + 3y^2 = 18$.

$g(x,y) = 2x^2 + 3y^2$

$\nabla f = \langle 2x+6, 12y \rangle$

$\lambda \nabla g = \langle 4\lambda x, 6\lambda y \rangle$

So $2x + 6 = 4\lambda x$ ①

$12y = 6\lambda y$ ②

and $2x^2 + 3y^2 = 18$ ③

From ②, $\lambda = 2$ or $y = 0$.

If $\lambda = 2$, then by ①

$2x + 6 = 8x$

$6x = 6$

$x = 1$

So by ③

$2 + 3y^2 = 18$

$y^2 = 16/3$

$y = \pm 4/\sqrt{3}$

If $y = 0$ then by ③

$2x^2 = 18$

$x^2 = 9$

$x = \pm 3$

This gives us several points to test:

$f(3,0) = 9 + 18 + 0 = 27$

$f(-3,0) = 9 - 18 + 0 = -9$

$f(1, \pm 4/\sqrt{3}) = 1 + 6 + 6(16/3) = 39$

$$\boxed{\text{So the maximum value is } 39 \text{ and the minimum value is } -9}$$

3. Evaluate the following integrals.

(a) $\int \sin 2x \cos^3 2x \, dx$

$u = \cos 2x$

$du = -2\sin 2x \, dx$

$-\frac{1}{2} du = \sin 2x \, dx$

U-substitution

so the integral becomes $\int u^3 (-\tfrac{1}{2}) du = -\frac{1}{2} \cdot \frac{1}{4} u^4 + C$

$= \boxed{-\frac{1}{8}\cos^4 2x + C}$

(b) $\int \frac{x}{\sqrt{9 - x^4}} \, dx$

u-substitution and inverse trig

$u = x^2$

$du = 2x \, dx$

$\frac{1}{2} du = x \, dx$

so the integral becomes $\int \frac{1}{\sqrt{9-u^2}} \cdot \frac{1}{2} du$

3 from $\sqrt{9}$

$= \frac{1}{2} \int \frac{1}{\sqrt{9(1 - u^2/9)}} du = \frac{1}{6} \int \frac{1}{\sqrt{1 - (\frac{u}{3})^2}} du =$

$\frac{1}{6} \sin^{-1}\left(\frac{u}{3}\right) \cdot 3 = \boxed{\frac{1}{2} \sin^{-1} \frac{x^2}{3} + C}$

chain rule

(c) $\int x \sec^2 x \, dx$

integration by parts

$u = x \qquad v = \tan x$

$du = 1 \, dx \qquad dv = \sec^2 x \, dx$

So the integral becomes

$uv - \int v \, du = x\tan x - \int \tan x \, dx = \boxed{x \tan x - \ln|\sec x| + C}$

(d) $\displaystyle\int_{\frac{4}{\sqrt{3}}}^{4} \frac{\sqrt{x^2-4}}{x}\,dx$ trig substitution

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta\,d\theta$

When $\underline{x=4}$, $\sec\theta = 2$, $\cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$

When $\underline{x = 4/\sqrt{3}}$, $\sec\theta = 2/\sqrt{3}$, $\cos\theta = \frac{\sqrt{3}}{2}$, $\theta = \pi/6$

So the integral becomes

$$\int_{\pi/6}^{\pi/3} \frac{\overset{\sqrt{x^2-4}}{\sqrt{4\sec^2\theta - 4}}}{\underbrace{\cancel{2\sec}\theta}_{x}} \cdot \underbrace{\cancel{2\sec}\theta\tan\theta\,d\theta}_{dx}$$

Since $\sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$

we get

$$\int_{\pi/6}^{\pi/3} 2\tan\theta \cdot \tan\theta\,d\theta = \int_{\pi/6}^{\pi/3} 2\tan^2\theta\,d\theta = \int_{\pi/6}^{\pi/3} 2(\sec^2\theta - 1)\,d\theta$$

$$= 2(\tan\theta - \theta)\Big]_{\pi/6}^{\pi/3} = 2(\tan\pi/3 - \pi/3) - 2(\tan\pi/6 - \pi/6)$$

$$= \boxed{2(\sqrt{3} - \pi/3) - 2(1/\sqrt{3} - \pi/6)}$$

(e) $\displaystyle\int \tan^3 x\ \sec^3 x\ dx$

derivative of sec x

$$= \int \tan^2 x\,\sec^2 x\,(\tan x\,\sec x)\,dx$$

Since $\tan^2 x = \sec^2 x - 1$ We have

$$\int (\sec^2 x - 1)\cdot\sec^2 x\cdot(\tan x\cdot\sec x)\,dx$$

$$= \int (\sec^4 x - \sec^2 x)\cdot(\tan x\cdot\sec x)\,dx$$

$u = \sec x$

$du = \tan x \cdot \sec x\,dx$

$$= \int u^4 - u^2\,du = \frac{u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$$

(f) $\displaystyle\int \frac{x^3+3x-2}{x^2-x}\,dx$ partial fractions

First, long division

$$\begin{array}{r} x+1 \\ x^2-x \,\big)\, \overline{x^3+3x-2} \\ -(x^3-x^2) \\ \hline x^2+3x \\ -(x^2-x) \\ \hline 4x-2 \end{array}$$

So $\dfrac{x^3+3x-2}{x^2-x} = x+1+\dfrac{4x-2}{x^2-x}$

Hence $\displaystyle\int \frac{x^3+3x-2}{x^2-x}\,dx = \int x+1+\frac{4x-2}{x^2-x}\,dx = \frac{x^2}{2}+x+\int \frac{4x-2}{x^2-x}\,dx$

$\dfrac{4x-2}{x(x-1)} = \dfrac{A}{x}+\dfrac{B}{x-1}$

So $4x-2 = A(x-1)+Bx$

when $x=1$ we get

$2=B$

when $x=0$ we get

$-2=-A$

$A=2$

So $\displaystyle\int \frac{x^3+3x-2}{x^2-x}\,dx = \frac{x^2}{2}+x+\int \frac{2}{x}+\frac{2}{x-1}\,dx$

$= \boxed{\dfrac{x^2}{2}+x+2\ln|x|+2\ln|x-1|+C}$

(g) (see next page)

(h) $\displaystyle\int \frac{4x^2-5x-15}{x^3-4x^2-5x}\,dx$ partial fractions

$x^3-4x^2-5x = x(x^2-4x-5) = x(x-5)(x+1)$

so $\dfrac{4x^2-5x-15}{x(x-5)(x+1)} = \dfrac{A}{x}+\dfrac{B}{x-5}+\dfrac{C}{x+1}$

or $4x^2-5x-15 = A(x-5)(x+1)+Bx(x+1)+Cx(x-5)$

When $x=0$ we get $-15 = A(-5)\cdot 1$ or $A=3$

When $x=5$ we get $4\cdot 25-25-15 = B\,5\cdot 6$

$60 = 30B$ or $B=2$

When $x=-1$ we get $4+5-15 = C(-1)(-6)$

$-6 = 6C$ or $C=-1$

So $\displaystyle\int \frac{4x^2-5x-15}{x^3-4x^2-5x}\,dx = \int \frac{3}{x}+\frac{2}{x-5}-\frac{1}{x+1}\,dx$

$= \boxed{3\ln|x|+2\ln|x-5|-\ln|x+1|+C}$

(g) $\int \sin^4 2x \ \cos^2 2x \ dx$

Since both sin and cos are to even powers, we need to use the half angle formula.

$$\cos^2 2x = \frac{1+\cos 4x}{2}$$

$$\sin^4 2x = (\sin^2 2x)^2 = \left(\frac{1-\cos 4x}{2}\right)^2 = \frac{1-2\cos 4x+\cos^2 4x}{4}$$

So the integral becomes

$$\int \left(\frac{1-2\cos 4x+\cos^2 4x}{4}\right)\cdot\left(\frac{1+\cos 4x}{2}\right)dx =$$

$$\frac{1}{8}\int 1-2\cancel{\cos 4x}+\cancel{\cos^2 4x}+\cancel{\cos 4x}-2\cancel{\cos^2 4x}+\cos^3 4x \, dx =$$

$$\frac{1}{8}\int 1-\cos 4x-\cos^2 4x+\cos^3 4x \, dx =$$

$$\frac{1}{8}\left(x-\frac{1}{4}\sin 4x-\int\cos^2 4x\,dx+\int\cos^3 4x\,dx\right) =$$

We compute the last two integrals.

$$\int\cos^2 4x\,dx = \frac{1}{4}\int\cos^2 u\,du = \frac{1}{4}\left(\frac{1}{2}u+\frac{1}{4}\sin 2u\right)+C = \frac{1}{8}\cdot 4x+\frac{1}{16}\sin 8x$$

$u = 4x$

$du = 4dx$

$$\int\cos^3 4x\,dx = \int(\cos 4x)(\cos^2 4x)\,dx = \int(\cos 4x)(1-\sin^2 4x)\,dx$$

$$= \int\cos 4x\,dx - \int\sin^2 4x\cos 4x\,dx$$

$u=4x \quad du = 4dx$

$$= \frac{1}{4}\sin 4x - \frac{1}{4}\int\sin^2 u\cos u\,du$$

$$= \frac{1}{4}\sin 4x - \frac{1}{4}\frac{\sin^3 4x}{3}+C$$

Putting it all together

$$\boxed{\frac{1}{8}\left(x-\frac{1}{4}\sin 4x-\left(\frac{1}{2}x+\frac{1}{16}\sin 8x\right)+\frac{1}{4}\sin 4x-\frac{1}{12}\sin^3 4x\right)+C}$$

vertical asymptote @ $x=3$.

(i) $\int_1^3 \frac{1}{\sqrt{3-x}}\,dx \quad = \lim_{t\to 3^-} \int_1^t \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-} -2\sqrt{3-x}\,\Big|_1^t$

(usub with $u=3-x$ $du=-x\,dx$

$$= \lim_{t\to 3^-} \Big(\underbrace{-2\sqrt{3-t}}_{\to 0 \text{ as}} + 2\sqrt{2}\Big) = \boxed{2\sqrt{2}}$$

$t\to 3^-$, i.e. t slightly smaller than 3.

u-substitution

(j) $\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{t\to -\infty} \int_t^0 xe^{-x^2}\,dx$

$u=-x^2$

$du=-2x\,dx \qquad -\frac{1}{2}du = x\,dx$

$$\int -\frac{1}{2}e^u\,du = -\frac{1}{2}e^u$$

$$= \underbrace{-\frac{1}{2}e^{-x^2}}_{\text{plug back in to limit statement}}$$

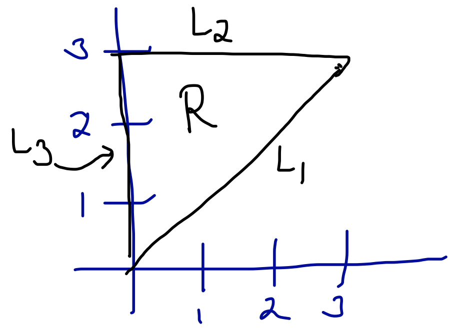
$$= \lim_{t\to -\infty} -\frac{1}{2}e^{-x^2}\Big|_t^0 =$$

$$\lim_{t\to -\infty} -\frac{1}{2}e^0 - \left(-\frac{1}{2}e^{-t^2}\right) = \lim_{t\to -\infty} \frac{1}{\underbrace{2e^{t^2}}_{}} - \frac{1}{2}$$

goes to 0 as $t\to -\infty$

So limit is $\boxed{\frac{-1}{2}}$

4. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ on the region defined by $x \geq 0$, $y \leq 3$, and $y \geq x$.



We first find local max/min inside R.

$\frac{\partial f}{\partial x} = 2x - 2 = 0$ when $x = 1$

$\frac{\partial f}{\partial y} = 2y - 4 = 0$ when $y = 2$

Hence a critical point at $(1,2)$ and $f(1,2) = 1 + 4 - 2 - 8 = -5$

Now check L_1. Along this line $y = x$ for $0 \leq x \leq 3$.

So $f(x,x) = 2x^2 - 6x$. Call $g(x) = 2x^2 - 6x$ then $g'(x) = 4x - 6$

$4x - 6 = 0$ if $x = 3/2$

$- - - \mid + + + +$ at $3/2$, so $x = 3/2$ is min

max occurs at endpoint $x = 3$

So we record max and min values

$f(3,3) = 9 + 9 - 6 - 12 = 0$

$f(3/2, 3/2) = 9/4 + 9/4 - 3 - 6 = -9/2$

Next check L_2. Here $y = 3$ for $0 \leq x \leq 3$. So $f(x,3) = x^2 + 9 - 2x - 12 = x^2 - 2x - 3$

Call $g(x) = x^2 + 2x - 3$, $g'(x) = 2x - 2$. This is 0 when $x = 1$

$- - - \mid + + + +$ at 1 → min here so max at endpoint $x = 3$

We record max/min values.

$f(1,3) = 1 + 9 - 2 - 12 = -4$

$f(3,3)$ done before

Finally check L_3. Here $x = 0$ where $0 \leq y \leq 3$. So $f(0,y) = y^2 - 4y$

Call $g(y) = y^2 - 4y$ and $g'(y) = 2y - 4$ This is 0 when $y = 2$

$- - - \mid + + +$ at 2 → min and max at $y = 0$.

We record max/min values.

$f(0,0) = 0$

$f(0,2) = 4 - 8 = -4$

Putting this all together we get $\boxed{\text{a max value of } 0,\ \text{a min value of } -5}$