Exam 2 Review Solutions

1. Find equations for the tangent line to the curve $\mathbf{r}(t) = \left\langle 4 - t^3, e^{-t}, \dfrac{1}{t+1} \right\rangle$ at $t = 3$.

First we find $\vec{r}\,'(t) = \left\langle -3t^2, -e^{-t}, \frac{-1}{(t+1)^2} \right\rangle$.

Second, notice that $\vec{r}(3) = \left\langle 4-27, e^{-3}, \frac{1}{4} \right\rangle = \left\langle -23, e^{-3}, \frac{1}{4} \right\rangle$

and $\vec{r}\,'(3) = \left\langle -27, -e^{-3}, \frac{-1}{16} \right\rangle$

So we use the vector equation of a line to get

$$\left\langle -23, e^{-3}, \tfrac{1}{4} \right\rangle + t \cdot \left\langle -27, -e^{-3}, \tfrac{-1}{16} \right\rangle$$

$$= \left\langle -23-27t, e^{-3} - te^{-3}, \tfrac{1}{4} - \tfrac{1}{16}t \right\rangle$$

$$= \left\langle -23-27t, e^{-3}(1-t), \tfrac{1}{4}\left(1 - \tfrac{t}{4}\right) \right\rangle$$

This gives the parametric equations

$$\boxed{x = -23-27t, \quad y = e^{-3}(1-t), \quad z = \tfrac{1}{4}\left(1 - \tfrac{t}{4}\right)}$$

2. Given the parametric equations $x = t^2 - 9$ and $y = t^2 - 8t$
(a) Find where the tangent is horizontal or vertical.

We first rewrite these parametric equations as a vector valued function, so $\vec{r}(t) = \langle t^2 - 9, t^2 - 8t \rangle$. Then any questions about tangent lines will require the derivative. So

$$\vec{r}\,'(t) = \langle 2t, 2t - 8 \rangle$$

Vertical tangent lines occur when there is no change in the x direction, so when $2t = 0$ or $\boxed{t = 0}$

horizontal tangent lines occur when there is no change in the y direction, so when $2t - 8 = 0$ or $\boxed{t = 4}$

(b) Find the equation of the tangent line at $t = 4$.

When $t = 4$, $\vec{r}(4) = \langle 16 - 9, 16 - 32 \rangle = \langle 7, -16 \rangle$

$\vec{r}\,'(4) = \langle 8, 0 \rangle$

So this line has corresponding vector $\langle 8, 0 \rangle$ and point $(7, -16)$

using the vector equation for a line we get

$$\langle 7, -16 \rangle + \langle 8, 0 \rangle t$$
$$= \boxed{\langle 7 + 8t, -16 \rangle}$$

Alternatively, a horizontal line at $t = 4$ must go through the point $(7, -16)$ so the line is $\boxed{y = -16.}$

3. Compute all the first and second partial derivatives of the following functions.

(a) $f(x, y) = x \ln(x^2 y) - 3y$

$$f_x(x,y) = x \cdot \frac{1}{x^2y} \cdot 2xy + \ln(x^2y) = \boxed{2+\ln(x^2y)}$$

$$f_y(x,y) = x \cdot \frac{1}{x^2y} \cdot x^2 - 3 = \boxed{\frac{x}{y} - 3}$$

$$f_{xx} = \frac{1}{x^2y} \cdot 2xy = \boxed{\frac{2}{x}}$$

$$f_{xy} = f_{yx} = \boxed{\frac{1}{y}}$$

$$f_{yy} = \boxed{\frac{-x}{y^2}}$$

(b) $f(x, y) = e^{\sqrt{x^2+y^2}}$

$$f_x(x,y) = \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}x\, e^{\sqrt{x^2+y^2}} = \boxed{\frac{x e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}}$$

$$f_y(x,y) = \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}y\, e^{\sqrt{x^2+y^2}} = \boxed{\frac{y e^{\sqrt{x^2+y^2}}}{\sqrt{y^2+y^2}}}$$

$$\boxed{\frac{e^{\sqrt{x^2+y^2}}\left(x^2+\sqrt{x^2+y^2}-x^2\sqrt{x^2+y^2}\right)}{x^2+y^2}}$$

$$f_{xx} = \frac{\sqrt{x^2+y^2}\left(x\frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}x\, e^{\sqrt{x^2+y^2}} + e^{\sqrt{x^2+y^2}}\right) - \frac{1}{\cancel{2}}\sqrt{x^2+y^2} \cdot \cancel{2}x \cdot x \cdot e^{\sqrt{x^2+y^2}}}{\left(\sqrt{x^2+y^2}\right)^2}$$

Since x any y are symmetric we know $f_{yy} = \boxed{\frac{e^{\sqrt{x^2+y^2}}\left(y^2+\sqrt{x^2+y^2}-y^2\sqrt{x^2+y^2}\right)}{x^2+y^2}}$

$$f_{xy} = f_{yx} = x\left(\frac{\cancel{\sqrt{x^2+y^2}} \cdot \frac{1}{\cancel{2}}\cancel{(x^2+y^2)^{-1/2}}\, \cancel{2}y\, e^{\sqrt{x^2+y^2}} - e^{\sqrt{x^2+y^2}}\, \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}y}{\left(\sqrt{x^2+y^2}\right)^2}\right) = \boxed{\frac{xy\, e^{\sqrt{x^2+y^2}}\left(1-\frac{1}{\sqrt{x^2+y^2}}\right)}{x^2+y^2}}$$

4. Compute the gradient for the function $f(x, y) = \cos(x^2 + y)$.

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$= \boxed{\langle -\sin(x^2+y) \cdot 2x, -\sin(x^2+y) \rangle}$$

5. Find an equation of the tangent plane of the function $f(x,y) = \dfrac{x}{\sqrt{y}}$ at (4,4).

$$f_x(x,y) = \frac{1}{\sqrt{y}} \qquad f_x(4,4) = \frac{1}{2}$$

$$f_y(x,y) = -\frac{1}{2}xy^{-3/2} = \frac{-x}{2\sqrt{y^3}} \qquad f_y(4,4) = \frac{-4}{2\cdot 8} = -\frac{1}{4}$$

$$f(4,4) = 2$$

Putting this together, we get

$$\boxed{z = 2 + \frac{1}{2}(x-4) - \frac{1}{4}(y-4)}$$

6. (a) Use the chain rule to find $\dfrac{df}{dt}$ when $f(x,y) = \ln x + \ln y$, $x = \cos t$, and $y = t^2$.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{1}{x}\cdot(-\sin t) + \frac{1}{y}\cdot 2t$$

$$= \frac{1}{\cos t}(-\sin t) + \frac{1}{t^2}\cdot 2t = \boxed{-\tan t + \frac{2}{t}}$$

(b) Use the chain rule to find $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$ where $f(x,y) = x^2 + \sin(xy)$, $x = e^{s+t}$, and $y = s + t$.

We need 4 pieces of information:

$$\frac{\partial f}{\partial x} = 2x + y\cos(xy) \qquad \frac{\partial x}{\partial s} = e^{s+t} \qquad \frac{\partial y}{\partial s} = 1$$

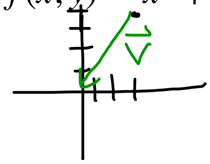
$$\frac{\partial f}{\partial y} = x\cos(xy) \qquad \frac{\partial x}{\partial t} = e^{s+t} \qquad \frac{\partial y}{\partial t} = 1$$

So $\dfrac{\partial f}{\partial s} = (2x + y\cos(xy))e^{s+t} + x\cos(xy)\cdot 1$

$$\boxed{= 2e^{s+t} + (s+t)\cos(e^{s+t}(s+t))e^{s+t} + e^{s+t}\cos(e^{s+t}(s+t))}$$

In fact, since $\dfrac{\partial x}{\partial s} = \dfrac{\partial x}{\partial t}$ and $\dfrac{\partial y}{\partial s} = \dfrac{\partial y}{\partial t}$ what we wrote above is also $\boxed{\dfrac{\partial f}{\partial t}}$

7. (a) Find the directional derivative of $f(x,y) = x^2 + 4y^2$ at the point (3,4) in the direction pointing toward the origin.



First, what is $\vec{u}$?

Points in direction of green

$\vec{v} = \langle -3, -4 \rangle$ But we need the unit vector. $|\vec{v}| = 5$

So $\vec{u} = \langle \frac{-3}{5}, \frac{-4}{5} \rangle$.

Second, what is $\nabla_f (3,4)$?

$\nabla_f = \langle 2x, 8y \rangle$ hence $\nabla_f(3,4) = \langle 6, 32 \rangle$

Then $D_{\vec{u}} f = \nabla_f \cdot \vec{u}$

$= \langle 6, 32 \rangle \cdot \langle \frac{-3}{5}, \frac{-4}{5} \rangle$

$= \frac{-18}{5} - \frac{128}{5} = \boxed{\frac{-146}{5}}$

(b) Is this function increasing or decreasing at the point (3,4) in the direction pointing toward the origin?

Since the value we found in (a) is negative, the rate of change is negative and so the function is $\boxed{\text{decreasing.}}$

8. Find the linearization $L(x,y)$ of $f(x,y) = x^2y^3$ at the point (2,1).

$f_x(x,y) = 2xy^3$ $\quad f_x(2,1) = 2 \cdot 2 \cdot 1 = 4$

$f_y(x,y) = 3x^2y^2$ $\quad f_y(2,1) = 3 \cdot 4 \cdot 1 = 12$

$f(2,1) = 4$

Putting everything together gives

$$L(x,y) = f_x(2,1)(x-2) + f_y(2,1)(x-1) + f(2,1)$$

$$= \boxed{4(x-2) + 12(x-1) + 4}$$