Final Exam Review Solutions

(1) Find the integrals.

(a) $\int x e^{5x+2}\, dx$

Integration by Parts

$u = x$ $\quad v = \frac{1}{5}e^{5x+2}$

$du = 1\,dx$ $\quad dv = e^{5x+2}\,dx$ can integrate using u-sub with $u = 5x+2$

$$\int x e^{5x+2} = \underbrace{\frac{1}{5}xe^{5x+2}}_{uv} \underbrace{- \int \frac{1}{5}e^{5x+2}\,dx}_{-\int v\,du} = \boxed{\frac{1}{5}xe^{5x+2} - \frac{1}{25}e^{5x+2} + C}$$

(b) $\int_1^4 \frac{e^{2x} - e^{4x}}{e^x}\, dx$ Notice that $e^{2x} = (e^x)^2$ and $e^{4x} = (e^x)^4$ so

$$= \int_1^4 \frac{(e^x)^2}{e^x} - \frac{(e^x)^4}{e^x}\,dx = \int_1^4 e^x - e^{3x}\,dx = e^x - \frac{1}{3}e^{3x}\Big]_1^4 =$$

$$\boxed{e^4 - \frac{1}{3}e^{12} - \left(e - \frac{1}{3}e^3\right)}$$

(c) $\int \frac{\ln(\cos^{-1} x)}{(\cos^{-1} x)\sqrt{1-x^2}}\, dx$ u-substitution

$u = \ln \cos^{-1} x$

$du = \frac{1}{\cos^{-1} x} \cdot \frac{-1}{\sqrt{1-x^2}}\,dx$ or $-du = \frac{1}{(\cos^{-1} x)\sqrt{1-x^2}}$

So the integral becomes

$$\int u\,(-1)\,du = \frac{-u^2}{2} = \boxed{\frac{-(\ln(\cos^{-1} x))^2}{2} + C}$$

(d) $\int x^2 \sin(x)\,\mathrm{d}x$ Integration by parts.

$u = x^2$ $\quad v = -\cos(x)$

$du = 2x\,dx$ $\quad dv = \sin(x)\,dx$

$= -x^2\cos(x) - \int -\cos(x) \cdot 2x\,dx$

$= -x^2\cos(x) + 2\int \cos(x) \cdot x\,dx$ Parts again $u = x$, $du = 1\,dx$, $v = \sin(x)$, $dv = \cos(x)\,dx$

$= -x^2\cos(x) + 2\left(x\sin(x) - \int \sin(x)\,dx\right)$

$= -x^2\cos(x) + 2x\sin(x) - 2(-\cos(x)) + C$

$= \boxed{-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C}$

(2) (a) Compute $\int_{-3}^{3} \frac{x^2 \sin x}{x^4+1}\,dx$. **Hint**: What is $f(-x)$?

sin is an odd function

$$f(-x) = \frac{(-x)^2\sin(-x)}{(-x)^4+1} = \frac{x^2(-\sin x)}{x^4+1} = -\frac{x^2\sin x}{x^4+1} = -f(x)$$

So this function is odd.

This means $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is continuous.

So this integral is $\boxed{0}$

(b) What goes wrong if we try to compute $\int_{-3}^{3} \frac{x^2 \sin x}{x^4-1}\,dx$?

Here, this function is not continuous on the interval we chose. We cannot compute the integral using the Fundamental Theorem of Calculus.

(3) Find the absolute maximum and minimum of the function $f(x) = x^2 - 8 \ln x$ on the interval [1,5].

We need to find local max or local min and compare to the endpoints.

$f'(x) = 2x - \frac{8}{x}$

$0 = 2x - \frac{8}{x}$

$2x = \frac{8}{x}$

$2x^2 = 8$

$x^2 = 4$

$x = \cancel{\pm} 2$

-2 is not in $[1,5]$

local $\rightarrow f(2) = 2^2 - 8\ln 2 \approx -1.545$

endpoints

$\rightarrow f(1) = 1 - 8\cancel{\ln 1}^{0} = 1$

$\rightarrow f(5) = 25 - 8\ln 5 \approx 12.124$

So absolute maximum of 12.124 at $x=5$
absolute minimum of -1.545 at $x=2$

(4) Find the equation of the tangent line to the graph $f(x) = 2e^{1-x^2}$ at the point (1,2).

We need a point on the line and the slope of $f(x)$ at that point.

Point is (1,2)

slope: $f'(x) = 2e^{1-x^2} \cdot (-2x)$ ← chain rule

$f'(1) = 2e^{1-1^2}(-2 \cdot 1) = 2 \cdot e^0 (-2) = -4$

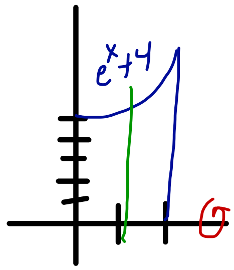
Apply Point-Slope Formula

$y - 2 = (-4)(x - 1)$

$y = -4x + 4 + 2$

$\boxed{y = -4x + 6}$

(5) Find the volume of the solid obtained by rotating the graph of $f(x) = e^x + 4$ from $x = 0$ to $x = 2$ about the x-axis.

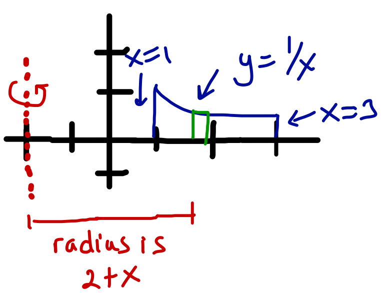


Slices are disks

$$\int_0^2 \pi (e^x+4)^2\, dx = \pi \int_0^2 e^{2x} + 8e^x + 16\, dx = \pi\left(\tfrac{1}{2}e^{2x} + 8e^x + 16x\right)\Big]_0^2 =$$

$$\boxed{\pi\left(\tfrac{1}{2}e^4 + 8e^2 + 32 - \left(\tfrac{1}{2} + 8\right)\right)}$$

(6) Find the volume of the solid obtained by rotating the region bounded by the graphs of $y = \dfrac{1}{x}$, $x = 1$, $x = 3$, and the x-axis about the line $x = -2$.



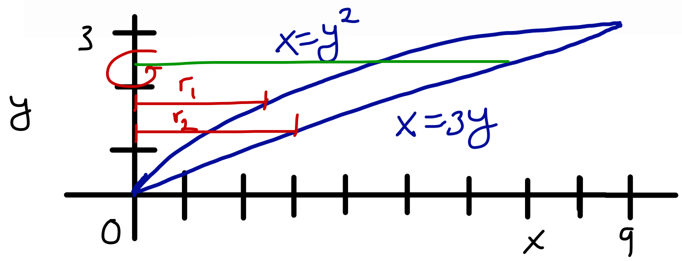
Shell

$$\int_1^3 2\pi \underbrace{(2+x)}_{\text{radius}} \underbrace{\tfrac{1}{x}}_{\text{height}}\, dx = 2\pi \int_1^3 \frac{2}{x} + 1\, dx =$$

$$2\pi\left(2\ln|x| + x\right)\Big]_1^3 = 2\pi\left(2\cdot\ln 3 + 3 - \left(\cancelto{0}{2\ln 1} + 1\right)\right) =$$

$$\boxed{2\pi(2\ln 3 + 2)}$$

(7) Find the volume of the solid obtained by rotating the region bounded by the graphs of the functions $x = y^2$ and $x = 3y$ about the y-axis.



First find the intersections

$y^2 = 3y$

$y^2 - 3y = 0$

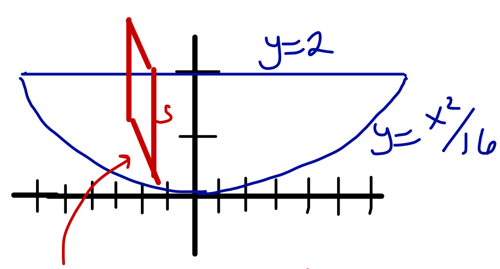
$y(y-3) = 0$ so $y=0$ & $y=3$

Washer

$$\int_0^3 \pi(r_2^2 - r_1^2)\,dy = \int_0^3 \pi\left((3y)^2 - (y^2)^2\right)dy = \pi\int_0^3 9y^2 - y^4\,dy =$$

$$\pi\left(3y^3 - \frac{y^5}{5}\right)\Big]_0^3 = \pi\left(3\cdot 3^3 - \frac{3^5}{5} - 0\right) = \boxed{\pi\left(3^4 - \frac{3^5}{5}\right)}$$

(8) A solid has as its base the region in the xy-plane bounded by the graphs of $x^2 = 16y$ and $y = 2$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a square.



Cross section square sticking up.

These intersect when $y=2$ or $x^2=32$ $x=\pm\sqrt{32}$

$x = \pm 4\sqrt{2} \approx \pm 5.66$

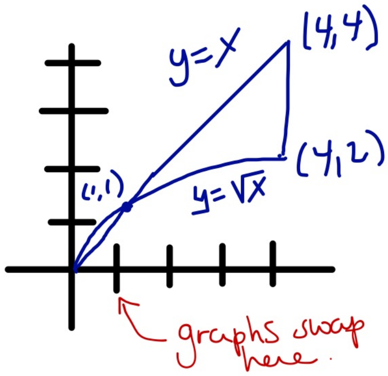
$S = 2 - x^2/16$ and is size of one side of cross section.

Area of cross section

$$\int_{-\sqrt{32}}^{\sqrt{32}} A(x)\,dx = \int_{-\sqrt{32}}^{\sqrt{32}} \left(2 - \frac{x^2}{16}\right)^2 dx = \int_{-\sqrt{32}}^{\sqrt{32}} 4 - \frac{x^2}{4} + \frac{x^4}{16^2}\,dx$$

$$= 4x - \frac{x^3}{12} + \frac{x^5}{5\cdot 16^2}\Big]_{-\sqrt{32}}^{\sqrt{32}} = \boxed{4\sqrt{32} - \frac{(\sqrt{32})^3}{12} + \frac{(\sqrt{32})^5}{5\cdot 16^2} - \left(-4\sqrt{32} - \frac{(-\sqrt{32})^3}{12} + \frac{(-\sqrt{32})^5}{5\cdot 16^2}\right)}$$

(9) Find the area of the region bounded by the graphs of $y = \sqrt{x}$ and $y = x$ from $x = 0$ to $x = 4$.



$$\int_0^1 \sqrt{x} - x\,dx + \int_1^4 x - \sqrt{x}\,dx$$

$$= \frac{2x^{3/2}}{3} - \frac{x^2}{2}\Big]_0^1 + \frac{x^2}{2} - \frac{2}{3}x^{3/2}\Big]_1^4 =$$

$$\frac{2}{3} - \frac{1}{2} - 0 + \frac{16}{2} - \frac{2}{3}\underbrace{(4)^{3/2}}_{=2^3=8} - \left(\frac{1}{2} - \frac{2}{3}\right) = \frac{4}{3} - 1 + 8 - \frac{16}{3} = 7 - \frac{12}{3}$$

$$= 3$$

(10) Find the average value of the function $f(x) = \dfrac{4(x^2+1)}{x^2}$ over the interval [1,3]. Explain where the formula for average value comes from.

The formula for average value is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

$$\frac{1}{3-1}\int_1^3 \frac{4(x^2+1)}{x^2}\,dx = \frac{1}{2}\int_1^3 4 + \frac{4}{x^2}\,dx$$

$$= \frac{1}{2}\left(4x - \frac{4}{x}\right)\Big]_1^3 = \frac{1}{2}\left(12 - \frac{4}{3} - \left(4 - \frac{4}{1}\right)\right) = \frac{1}{2}\left(12 - \frac{4}{3}\right) = \boxed{\frac{16}{3}}$$

We get average value by dividing the interval [a,b] into n pieces and approximating $\dfrac{\sum_{i=1}^{n} f(x_i)}{n}$ Then we take more & more intervals to get better approximations To get the actual average value, we take a limit $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \lim_{n\to\infty} \underbrace{\frac{\Delta x}{b-a}}_{=\frac{1}{n}} \sum_{i=1}^{n} f(x_i) =$ since $n = \dfrac{b-a}{\Delta x}$

$$\frac{1}{b-a}\underbrace{\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x}_{\text{definition of integral}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

(11) Find the limits.

(a) $\lim\limits_{x\to\infty} \dfrac{x^{\frac{3}{2}} + 5x - 4}{x \ln x}$

Since the limit of the top and bottom go to ∞ we can use L'Hôpital.

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{3}{2}x^{\frac{1}{2}} + 5}{\cancel{x}\cdot\frac{1}{\cancel{x}} + \ln x}$$

Limit is still ∞ on top and bottom so apply L'Hôpital again.

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{3}{4}x^{-\frac{1}{2}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{3}{4}\frac{x}{\sqrt{x}} = \lim_{x\to\infty} \frac{3}{4}\sqrt{x} = \infty$$

Algebra

(b) $\lim\limits_{x\to\infty} x(e^{\frac{2}{x}} - 1)$

$\lim\limits_{x\to\infty} x = \infty$ $\lim\limits_{x\to\infty} e^{2/x} - 1 = 0$ so apply L'Hôpital with $x = \frac{1}{1/x}$

$$= \lim_{x\to\infty} \frac{e^{2/x} - 1}{1/x} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{x\to\infty} \frac{e^{2/x}(\cancel{-}2\cancel{x^{-2}})}{\cancel{-x^{-2}}} = \lim_{x\to\infty} 2e^{2/x} = 2$$

Super Rule!

$2/x$ goes to 0 as $x\to\infty$ so $e^{2/x}$ goes to 1.