Exam 1 Review Solutions

(1) Find an equation of the tangent line to the curve $f(x) = 3x + 2x^2$ at the point $x = 1$.

Take a derivative to find the slope:

$f'(x) = 3 + 4x$ @ $x=1$ $f'(1) = 3 + 4 \cdot 1 = 7$

Find the point on the curve:

@ $x=1$, $f(1) = 3 \cdot 1 + 2 \cdot 1^2 = 5$

Use point slope formula:

$y - y_0 = m(x - x_0)$ or $\boxed{y - 5 = 7(x-1)}$

(2) Evaluate the limit, if it exists. Show or justify the steps you use.

(a) $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+1} - 1}{x}$

We get rid of the $\sqrt{\ }$ in the numerator.

$$\lim_{x\to 0} \left(\frac{\sqrt{x+1} - 1}{x}\right)\left(\frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}\right) = \lim_{x\to 0} \frac{(\sqrt{x+1})^2 - 1}{x(\sqrt{x+1} + 1)} =$$

$$\lim_{x\to 0} \frac{x + \cancel{1} - \cancel{1}}{x(\sqrt{x+1} + 1)} = \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+1} + 1)} \overset{\text{OPOT}}{=} \lim_{x\to 0} \frac{1}{\sqrt{x+1} + 1} =$$

Since continuous @ $x = 0$.

$$\frac{1}{\sqrt{1} + 1} = \boxed{\frac{1}{2}}$$

(b) $\displaystyle\lim_{h\to 2} \frac{h^2 - 3h - 4}{2h - 1}$

Since the denominator is not 0 @ $h=2$ and since this is a rational function, we can use DSP:

$$\lim_{h\to 2} \frac{h^2 - 3h - 4}{2h - 1} = \frac{2^2 - 3 \cdot 2 - 4}{2 \cdot 2 - 1} = \frac{-6}{3} = \boxed{-2}$$

(c) $\lim_{x\to-3} \frac{|3-x|}{x-3}$

By the definition of absolute value $\frac{|3-x|}{x-3} = \begin{cases} \frac{3-x}{x-3} & \text{if } 3-x\geq 0 \text{ or } x\leq 3 \\ \frac{-(3-x)}{x-3} & \text{if } 3-x<0 \text{ or } x>3 \end{cases}$

So near $x=-3$, this function is

$\frac{3-x}{x-3} = \frac{-(x-3)}{x-3} = -1$. Hence $\boxed{\lim_{x\to-3} \frac{|3-x|}{x-3} = -1}$

If we had asked for $\lim_{x\to 3} \frac{|3-x|}{x-3}$ then the limit would be undefined.

(3) Evaluate the limit of $\lim_{x\to 3} \sqrt{\frac{4x-3+x^2}{2x^2+x+1}}$ and justify each step by indicating the appropriate Limit Laws.

$$\lim_{x\to 3}\sqrt{\frac{4x-3+x^2}{2x^2+x+1}} \overset{11}{=} \sqrt{\lim_{x\to 3}\frac{4x-3+x^2}{2x^2+x+1}} \overset{5}{=} \sqrt{\frac{\lim_{x\to 3} 4x-3+x^2}{\lim_{x\to 3} 2x^2+x+1}}$$

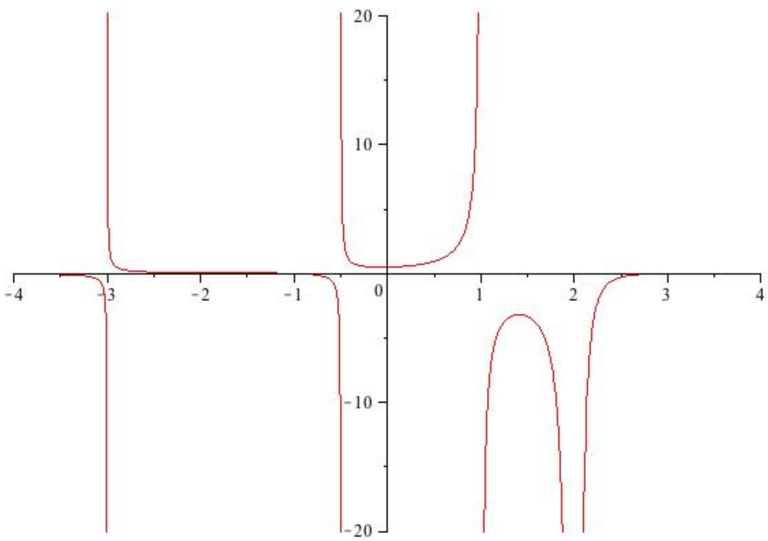
$$\overset{1 \text{ or } 2}{=} \sqrt{\frac{\lim_{x\to 3} 4x - \lim_{x\to 3} 3 + \lim_{x\to 3} x^2}{\lim_{x\to 3} 2x^2 + \lim_{x\to 3} x + \lim_{x\to 3} 1}} \overset{3}{=} \sqrt{\frac{4\lim_{x\to 3} 3 - 3 + (\lim_{x\to 3} x)^2}{2(\lim_{x\to 3} x)^2 + \lim_{x\to 3} x + 1}} \quad 7 \quad 6$$

$$\overset{8}{=} \sqrt{\frac{4\cdot 3-3+9}{2\cdot 9+3+1}} = \sqrt{\frac{18}{22}} = \sqrt{\frac{9}{11}} = \frac{3}{\sqrt{11}}$$

(4) Below is the graph of a function $f(x)$. State the following.

(a) $\lim_{x \to -3^-} f(x) = -\infty$ (c) $\lim_{x \to 1} f(x)$ undefined since one sided limits are different

(b) $\lim_{x \to -3^+} f(x) = \infty$ (d) $\lim_{x \to 2} f(x)$ $-\infty$



(5) Prove the following statement using the ε, δ definition of limit.

$$\lim_{x \to 3} 3x + 2 = 11$$

If we're given an $\varepsilon > 0$ with

$$|f(x) - L| < \varepsilon$$
$$|3x + 2 - 11| < \varepsilon$$
$$|3x - 9| < \varepsilon$$

Factor out a 3.

$$3|x - 3| < \varepsilon$$
$$|x - 3| < \varepsilon/3$$

So we should pick $\delta = \varepsilon/3$ so that

$$0 < |x - a| < \delta.$$

(6) Use the formal definition of the derivative to find $f'(x)$ for the function $f(x) = \dfrac{1}{2x+1}$.

plug in

$$\text{We know } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{2(x+h)+1} + \frac{1}{2x+1}}{h}$$

common denominator

multiply out numerator

$$= \lim_{h \to 0} \frac{\frac{2x+1-2(x+h)-1}{(2x+1)(2(x+h)+1)}}{h} = \lim_{h \to 0} \frac{\cancel{2x}+\cancel{1}-\cancel{2x}-2h-\cancel{1}}{h(2x+1)(2(x+h)+1)}$$

cancel

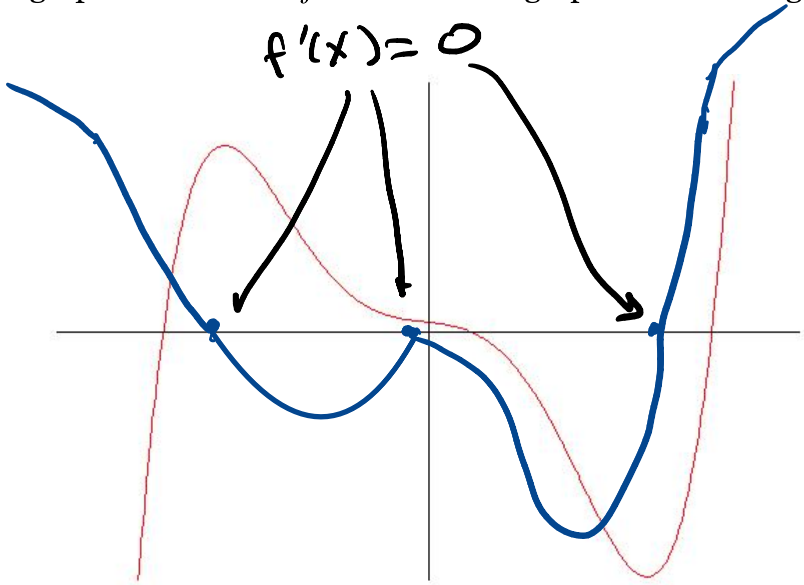
OPOT

$$= \lim_{h \to 0} \frac{-2\cancel{h}}{\cancel{h}(2x+1)(2(x+h)+1)} = \lim_{h \to 0} \frac{-2}{(2x+1)(2(x+h)+1)}$$

DSP

$$= \frac{-2}{(2x+1)(2x+1)} = \frac{-2}{(2x+1)^2}$$

(7) Below is the graph of a function f. On the same graph sketch a rough graph of its derivative.



Blue is a rough sketch of f'(x)

(8) Let $f(r) = \dfrac{r^2 + 2r - 3}{r^2 + r - 6}$. $= \dfrac{(r+3)(r-1)}{(r+3)(r-2)}$ DSP

(a) Compute $\lim_{r \to -3} f(r)$.

$$\lim_{r \to -3} f(r) = \lim_{r \to -3} \frac{r-1}{r-2} = \frac{-3-1}{-3-2} = \frac{-4}{-5} = \boxed{\frac{4}{5}}$$

OPOT

(b) Determine the infinite limit $\lim_{r \to 2^-} f(r)$. Using blue in (a) we see that as $r \to 2$ from the left

$r+3$ is >0
$r-1$ is >0
$r-2$ is <0

so $\dfrac{(r+3)(r-1)}{(r+3)(r-1)}$ is $\dfrac{+\cdot+}{+\cdot-}$ hence $\boxed{-\infty}$

(9) Differentiate the following functions.

(a) $f(x) = 1 - 3x + 2x^3$

$$\boxed{f'(x) = -3 + 6x^2}$$

(b) $g(t) = 5t^2 - \dfrac{2}{t} = 5t^2 - 2t^{-1}$

$$f'(x) = 10t - 2(-1)t^{-2} = \boxed{10t + \frac{2}{t^2}}$$

(c) $f(x) = \dfrac{1+x}{x^2}$

$$f'(x) = \frac{x^2 \frac{d}{dx}(1+x) - (1+x)\frac{d}{dx}x^2}{(x^2)^2} = \boxed{\frac{x^2 - (1+x)2x}{x^4}}$$

(d) $f(x) = x \cdot \cos(x)$

$$f'(x) = x \frac{d}{dx}\cos(x) + \cos(x) \cdot \frac{d}{dx}x = x(-\sin(x)) + \cos(x)$$
$$= \boxed{-x\sin(x) + \cos(x)}$$