# Math 218: Combinatorics

Homework 17 : Due December 6

1. In each part of this problem, let $G = (V, E)$ where $V = [20]$ and $E$ is the edge set described below. For each part prove or disprove that (i) the graph is connected (if it is not connected, determine the connected components), (ii) the graph has an Euler trail, and (iii) the graph has a Hamiltonian path

   (a) $\{a, b\} \in E$ if and only if $a + b$ is **odd**.

   (b) $\{a, b\} \in E$ if and only if $a \times b$ is **even**.

2. Explain your answer to each part.

   (a) Which complete graphs $K_n$ have Eulerian tours?

   (b) Which complete graphs $K_n$ have Eulerian trails (but not tours)?

   (c) Which trees have an Eulerian trail?

   (d ) Which trees have a Hamiltonian path?

3. Explain your answer to both parts.

   (a) Find an example of a connected, simple graph with degree sequence $\{3, 3, \ldots, 3\}$ that does not have a Hamiltonian path.

   (b) Find an example of a simple graph $G$ of order $n \geq 3$ which does not have any Hamiltonian cycles and yet, for every pair of nonadjacent verices $x$ and $y$, deg $(x)$ + deg $(y) \geq n - 1$. (This shows that we cannot do any better than the Ore Property bound.)

4. Let $T$ be a tree of order $n > 1$. Prove that the number of leaves is

$$2 + \sum_{\deg\,(v_i) \geq 3} (\deg\,(v_i) - 2)$$

   where the sum is over all vertices of degree 3 or more.