ELLIPTIC FACTORS IN JACOBIANS OF HYPERELLIPTIC CURVES WITH CERTAIN AUTOMORPHISM GROUPS ERRATA

JENNIFER PAULHUS

1. Explanation of Error

In [2], we computed the decompositions of Jacobian varieties of hyperelliptic curves of genus up to 20 with certain automorphism groups. The technique used to decompose these Jacobian varieties is correct, however it requires computing the irreducible $\mathbb{Q}$-characters of certain groups. At this point in the technique, we made a small error which impacted the computations for a few groups. We erroneously computed irreducible $\mathbb{Q}$-characters by only determining the Galois conjugates of each $\mathbb{C}$-character and adding those together. This is not always correct, as the *Schur index* must also be considered. As a side note, other papers by the author using this technique are not impacted, as those papers study groups with characters of Schur index one only.

Definition. Let G be a finite group and χ an irreducible $\mathbb{C}$-character of G. Let $\mathbb{Q}(\chi)$ denote the finite extension of $\mathbb{Q}$ containing all the values of χ on elements of G. The **Schur index of** χ **over** $\mathbb{Q}$ is the smallest degree of all field extensions $[S : \mathbb{Q}(\chi)]$ where S varies over all fields such that $\mathbb{Q}(\chi) \subset S \subset \mathbb{C}$ and so that χ is realizable in S.

As mentioned above, the Schur index impacts the irreducible $\mathbb{Q}$-characters.

Proposition. [1, Exercise 70.30.2] *Let $\{\chi_1, \ldots, \chi_r\}$ be the irreducible $\mathbb{C}$-characters of a finite group G. Then θ is an irreducible $\mathbb{Q}$-character if and only if $\theta = m(\chi_i) \cdot (\chi_i + \chi_i^\sigma + \cdots)$ where the $\{\chi_i^\sigma\}$ are the distinct conjugates of χ_i, an irreducible $\mathbb{C}$-character of G.*

Of the groups considered in [2], the following four have character(s) with Schur index greater than one: $\mathrm{SL}_2(3)$, $\mathrm{SL}_2(5)$, W_2, and W_3. All other groups in the paper (and corresponding decomposition of Jacobians of curves with those automorphism groups) are unaffected. Below we modify the results of [2] to account for the proper computation of the irreducible $\mathbb{Q}$-characters.

2. Irreducible $\mathbb{Q}$-characters

For completeness, for each of the four groups impacted by this error (groups having at least one character with Schur index over $\mathbb{Q}$ greater than one) we present its table of irreducible $\mathbb{Q}$-characters here.

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Character	Conjugacy class order						
	1	2	3	3	4	6	6
χ_1	1	1	1	1	1	1	1
χ_2	2	2	−1	−1	2	−1	−1
χ_3	4	−4	−2	−2	0	2	2
χ_4	4	−4	1	1	0	−1	−1
χ_5	3	3	0	0	−1	0	0

TABLE 1. Irreducible $\mathbb{Q}$-characters for $\mathrm{SL}_2(3)$. The red character comes from an irreducible $\mathbb{C}$-character of Schur index 2.

Character	Conjugacy class order								
	1	2	3	4	5	5	6	10	10
χ_1	1	1	1	1	1	1	1	1	1
χ_2	8	−8	−4	0	−2	−2	4	2	2
χ_3	6	6	0	−2	1	1	0	1	1
χ_4	4	4	1	0	−1	−1	1	−1	−1
χ_5	8	−8	2	0	−2	−2	−2	2	2
χ_6	5	5	−1	1	0	0	−1	0	0
χ_7	12	−12	0	0	2	2	0	−2	−2

TABLE 2. Irreducible $\mathbb{Q}$-characters for $\mathrm{SL}_2(5)$. The red characters comes from irreducible $\mathbb{C}$-characters of Schur index 2.

Character	Conjugacy class order									
	1	2	2	2	3	4	4	4	4	6
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	−1	−1	−1	−1	1
χ_3	2	−2	2	−2	2	0	0	0	0	−2
χ_4	4	−4	4	−4	−2	0	0	0	0	2
χ_5	2	2	2	2	−1	0	0	0	0	−1
χ_6	3	3	−1	−1	0	1	1	−1	−1	0
χ_7	3	3	−1	−1	0	−1	−1	1	1	0
χ_8	6	−6	−2	2	0	0	0	0	0	0

TABLE 3. Irreducible $\mathbb{Q}$-characters for W_2. The red character comes from an irreducible $\mathbb{C}$-character of Schur index 2.

	Conjugacy class order							
Character	1	2	3	4	4	6	8	8
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	1	-1	-1
χ_3	2	2	-1	2	0	-1	0	0
χ_4	8	-8	-4	0	0	4	0	0
χ_5	3	3	0	-1	1	0	-1	-1
χ_6	3	3	0	-1	-1	0	1	1
χ_7	8	8	2	0	0	-2	0	0

TABLE 4. Irreducible $\mathbb{Q}$-characters for W_3. The red characters comes from irreducible $\mathbb{C}$-characters of Schur index 2.

3. Corrected Results

In this section we correct the results from [2]. First, Theorem 1 in the paper is not correct, as this curve has automorphism group $\mathrm{SL}_2(3)$. The theorem should state:

Theorem 1. *The hyperelliptic curve of genus* 4 *with affine model*

$$X: y^2 = x(x^4-1)(x^4+2\sqrt{-3}\,x^2+1)$$

has a Jacobian variety that decomposes as $E^2 \times A_2$ for an elliptic curve E and a dimension 2 *variety A_2.*

Next, Theorem 5 from the paper includes a table of results from our computations. The correct decompositions follow in Table 5 of this errata. The red entries are those that have changed due to the Schur index. For three of the groups in the table ($A_4 \times C_2$, $S_4 \times C_2$, and W_2), the groups may act in several different ways on the space of all curves of a fixed genus with that automorphism group. As such, it is now possible to find finer decompositions in some cases (or even slightly different decompositions depending on the action) but we have kept here the decompositions that were computed at the time the original paper was written. Also, we took this opportunity to fix a typo on the $g = 11$ example with automorphism group $S_4 \times C_2$.

Finally, in section 5 of the original paper, we computed the Jacobian variety decomposition for certain groups for arbitrarily large genus. However, these groups all have character(s) with Schur index greater than 1. The corrections for this section are below where, once more, modifications from the original paper are highlighted in red.

5.1: The group $\mathrm{SL}_2(3)$. Suppose X is a curve of genus g with automorphism group $\mathrm{SL}_2(3)$. Let $d = \lfloor (g-1)/6 \rfloor$ be as above. The computation of χ_V depends on the value of g modulo 6.

- Suppose $g \equiv 2 \bmod 6$. Applying the monodromy information given in Table 5 of the original paper to Equation (3) of the original paper yields

$$\chi_V = 2\chi_{\mathrm{triv}} + (d+1)\chi_{(1)} - \chi_{(4)} - 2\chi_{(3)} - d\chi_{(2)}.$$

Genus	Aut. group	Dimension	Jacobian decomposition
3	$S_4 \times C_2$	0	E^3
4	$\mathrm{SL}_2(3)$	0	$E_1^2 \times A_2$
5	$A_4 \times C_2$	1	$E^3 \times A_2$
	W_2	0	$E^3 \times A_2$
	$A_5 \times C_2$	0	E^5
6	$\mathrm{GL}_2(3)$	0	$E_1^2 \times E_2^4$
7	$A_4 \times C_2$	1	$E_1 \times E_2^3 \times E_3^3$
8	$\mathrm{SL}_2(3)$	1	$A_2^2 \times A_4$
	W_3	0	$A_2^2 \times A_4$
9	$A_4 \times C_2$	1	$E^3 \times A_2^3$
	W_2	0	$E \times A_{2,1} \times A_{2,2}^3$
	$A_5 \times C_2$	0	$E_1^4 \times E_2^5$
10	$\mathrm{SL}_2(3)$	1	$A_3^2 \times A_4$
11	$A_4 \times C_2$	2	$A_2 \times A_3^3$
	$S_4 \times C_2$	1	$E_1^2 \times E_2^3 \times A_2^3$
12	$\mathrm{SL}_2(3)$	1	$A_{4,1} \times A_{4,2}^2$
	W_3	0	$A_{4,1} \times A_{4,2}^2$
13	$A_4 \times C_2$	2	$E \times A_{3,1} \times A_{3,2}^3$
14	$\mathrm{SL}_2(3)$	2	$A_4^2 \times A_6$
	$\mathrm{GL}_2(3)$	1	$A_2^4 \times A_3^2$
	$\mathrm{SL}_2(5)$	0	$A_{2,1}^2 \times A_{2,2}^3 \times A_4$
15	$A_4 \times C_2$	2	$A_2^3 \times A_3^3$
	$S_4 \times C_2$	1	$E_1 \times E_2^2 \times A_{2,1}^3 \times A_{2,2}^3$
	$A_5 \times C_2$	0	$E_1^4 \times E_2^5 \times A_2^3$
16	$\mathrm{SL}_2(3)$	2	$A_5^2 \times A_6$
17	$A_4 \times C_2$	3	$E \times A_{4,1} \times A_{4,2}^3$
	W_2	1	$E \times A_{4,1} \times A_{4,2}^3$
18	$\mathrm{SL}_2(3)$	2	$A_{6,1} \times A_{6,2}^2$
	$\mathrm{GL}_2(3)$	1	$A_{3,1}^2 \times A_{3,2}^4$
19	$A_4 \times C_2$	3	$E \times A_2^3 \times A_4^3$
20	$\mathrm{SL}_2(3)$	3	$A_6^2 \times A_8$
	W_3	1	$A_6^2 \times A_8$
	$\mathrm{SL}_2(5)$	0	$A_2^2 \times A_{4,1} \times A_{4,2}^3$

Table 5. Corrected Jacobian variety decompositions. For each genus g and automorphism group G, we list the dimension of the moduli space of genus-g hyperelliptic curves with automorphism group G, along with a decomposition of the Jacobian of these curves. The notation is explained in Theorem 5 of the original paper.

Computing the inner product of each irreducible $\mathbb{Q}$-character (see Table 1) with χ_V gives $J_X \sim A_{2(d+1)} \times A_{2d}^2$.

- Suppose $g \equiv 4 \bmod 6$. Applying the monodromy information from Table 5 of the original paper, we find that
$$\chi_V = 2\chi_{\text{triv}} + (d+1)\chi_{(1)} - \chi_{(4)} - \chi_{(6)} - \chi_{(3)} - d\chi_{(2)}.$$
This gives $J_X \sim A_{2(d+1)} \times A_{2d+1}^2$.
- Finally, suppose $g \equiv 0 \bmod 6$. Using Table 5 of the original paper, we compute that
$$\chi_V = 2\chi_{\text{triv}} + (d+1)\chi_{(1)} - \chi_{(4)} - 2\chi_{(6)} - d\chi_{(2)}.$$
This gives $J_X \sim A_{2(d+1),1} \times A_{2(d+1),2}^2$.

5.2: The group $\mathrm{SL}_2(5)$**.** Computing the inner products of the irreducible $\mathbb{Q}$-characters (listed in earlier section) with χ_V (listed below for the four congruence classes of g) produces decompositions of the form $A_{4(d+1)} \times A_{2j}^2 \times A_{2k}^3$, where d, j, and k are determined by the congruence class of g modulo 30, and where $d = \lfloor (g-1)/30 \rfloor$ is the dimension of the family of curves with this automorphism group. Note that the values of j and k below are unchanged from the original paper.

- Suppose $g \equiv 14 \bmod 30$. Then the Hurwitz character is
$$\chi_V = 2\chi_{\text{triv}} + (d+1)\chi_{(1)} - \chi_{(4)} - \chi_{(3)} - \chi_{(5)} - d\chi_{(2)},$$
and we have $j = 2d+1$ and $k = 3d+1$.
- Suppose $g \equiv 20 \bmod 30$. Then the Hurwitz character is
$$\chi_V = 2\chi_{\text{triv}} + (d+1)\chi_{(1)} - \chi_{(4)} - \chi_{(3)} - \chi_{(10)} - d\chi_{(2)},$$
and we have $j = 2d+1$ and $k = 3d+2$.
- Suppose $g \equiv 24 \bmod 30$. Then the Hurwitz character is
$$\chi_V = 2\chi_{\text{triv}} + (d+1)\chi_{(1)} - \chi_{(4)} - \chi_{(6)} - \chi_{(5)} - d\chi_{(2)},$$
and we have $j = 2(d+1)$ and $k = 3d+2$.
- Finally, suppose $g \equiv 0 \bmod 30$. Then the Hurwitz character is
$$\chi_V = 2\chi_{\text{triv}} + (d+1)\chi_{(1)} - \chi_{(4)} - \chi_{(6)} - \chi_{(10)} - d\chi_{(2)},$$
so $j = 2(d+1)$ and $k = 3(d+1)$.

5.3: The group W_3**.** We compute the decomposition of the Jacobian in the two cases as follows, where $d = \lfloor (g-1)/12 \rfloor$

- When $g \equiv 8 \bmod 12$, the Hurwitz character is
$$\chi_V = 2\chi_{\text{triv}} + (d+1)\chi_{(1)} - \chi_{(4)} - \chi_{(3)} - \chi_{(8)} - d\chi_{(2)}$$
and $J_X \sim A_{4(d+1)} \times A_{2(2d+1)}^2$.
- When $g \equiv 0 \bmod 12$, the Hurwitz character is
$$\chi_V = 2\chi_{\text{triv}} + (d+1)\chi_{(1)} - \chi_{(4)} - \chi_{(6)} - \chi_{(8)} - d\chi_{(2)}$$
and $J_x = A_{4(d+1),1} \times A_{4(d+1),2}^2$.

References

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Department of Mathematics and Statistics, Grinnell College, Grinnell, IA 50112
E-mail address: `paulhusj@grinnell.edu`