Group Actions and Riemann Surfaces

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Abstract

Riemann surfaces are fascinating topological objects with extra structure which allows us to translate ideas from analysis onto them. Additionally, groups can describe ways to permute the elements of a Riemann surface through a process called a group action. To give you a sense of what my field of research entails, we first need to understand group actions and Riemann surfaces. While this may seem like a tall order, you can grasp the ideas of these topics with just a first semester undergraduate course in both algebra and analysis, along with a basic knowledge of the complex numbers (such as roots of unity). An undergraduate course in topology may aid your understanding, but I define key ideas along the way.

1 Introduction

Perhaps the most ubiquitous application of groups in modern mathematics is via group actions. We define group actions formally in the next section, but first I want to give you an idea of the topic. Given a set of objects, a group *acts on* the set if each element of the group can represent some shuffling of the set. Applications of this topic range from counting colorings up to some equivalence in combinatorics, to proving classical congruences in number theory, to studying automorphisms of surfaces. We will talk about the last topic later. Even physicists and chemists get in on the action (pun intended!) by, for example, interpreting problems about molecular structures or systems with rotational symmetries as problems of groups acting on vector spaces.

Here is a geometric example. Take your favorite regular polygon (regular means all sides are the same length and all angles are the same). My favorite is the regular pentagon. Place the shape on paper with one point (also called a *vertex*) at the top (north), label the vertices (five in my case), and mark where the vertices are located on the paper. Imagine lifting the shape up, moving it around without stretching or shrinking it, and then putting it back down to cover the same outline on the paper. In mathematical terms the valid rearrangements of the shape on paper are called *symmetries*. Pause for a minute and find some symmetries of your favorite polygon.

Check out Figure 1 for some examples. One symmetry is rotating the shape clockwise so that each vertex moves to the one to the right of it. We could of course rotate again, and again, and again, but once we rotate five times, the object is back to its original positioning. Notice that rotating once counterclockwise has the same outcome as rotating clockwise 4 times. In Figure 1, we also show reflection across the line from the vertex at the top to the middle of the bottom edge (or from the vertex at the top to the vertex at the bottom if your favorite regular polygon has an even number of vertices).

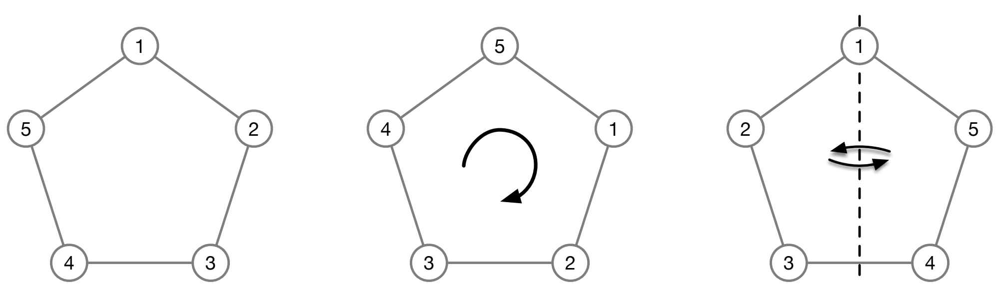

Figure 1: From left to right: the regular pentagon as we describe positioned on paper, rotation by 72°, and reflection.

To create more symmetries we can compose two known symmetries, thus forming a group with operation composition. You may recognize this group from abstract algebra: the *dihedral group*. There is some disagreement on

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how to denote this group. I call the group of symmetries on the regular n-gon D_n with $|D_n| = 2n$, but be warned that some books and research papers call the same group D_{2n}. This is what passes as drama in mathematics! Each symmetry (that is each element of the dihedral group) describes a way to shuffle the vertices of an n-gon, and so we come to our first example of a group action: D_n acts on the set of vertices of an n-gon.

In your linear algebra class you were introduced to linear transformations and a handy way of representing them: matrices. What are linear transformations? Functions from vectors to vectors (with some important additional properties, i.e. they preserve addition and scalar multiplication). In the case of $\mathbb{R}^2$, what does applying a linear transformation do? It moves vectors around the Cartesian plane. As an example, take the matrix $-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. It sends each vector to the corresponding vector in the opposite direction, rotating all vectors by $180°$, except the zero vector which it fixes. What about a matrix like $M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$? It stretches a vector by a multiple of 2 in the x direction and by a multiple of 3 in the y direction so each vector will get sent to one that has twice the x component and three times the y component, again fixing the zero vector.

Consider the linear transformation represented by $M = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$. Think for a moment about what happens to vectors along the line $y = x$. They are fixed precisely because $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvalue with eigenvector 1. What is the other eigenvalue and what does it tell you about how the linear transformation moves the corresponding eigenvectors? To describe what M does to the remaining (non-eigenvector) vectors, choose a basis of $\mathbb{R}^2$ consisting of one eigenvector for each of the two eigenvalues and rewrite each vector in terms of that basis. The linear transformation will stretch the original vector by the amount of the component corresponding to the first eigenvector, and stretch it by five times the amount of the component corresponding to the second eigenvector. (This should give you a hint for what the other eigenvalue is!)

And so we have our second example of a group action. The set of all 2×2 matrices with entries in $\mathbb{R}$ describes a way to shuffle the vectors in $\mathbb{R}^2$. You might be a bit worried because the set of all 2×2 matrices with entries in $\mathbb{R}$ does not form a group under the operation of matrix multiplication (which axiom(s) fail?). We will address your concerns in the next section.

My goal in this paper is to introduce you to special group actions which I use in my own research. The sets are topological objects called *Riemann surfaces* (named for German mathematician Bernhard Riemann who first defined them) and there are many different groups which can act on these sets. In order to understand this topic, we need to first understand group actions (Section 3) and Riemann surfaces (Section 4). In Section 5, I describe what it means for groups to act on Riemann surfaces, and then connect this to my own research (current and future) in Sections 6 and 7. In an attempt to keep the exposition from getting cluttered, precise definitions not essential to a first understanding of the subject are in their own section (8). Throughout the paper, I prompt you to pause and consider some questions. If this is your first time grappling with these topics, I highly recommend you stop reading at these points, and think for a bit (probably with paper and pencil).

Finally, here is some standard notation I use: G is a group and 1_G its identity element, $|G|$ is the cardinality of G (size as a set), and $o(g)$ is the order of an element $g \in G$. For a complex number $z = x + yi$, Re(z) is x, the real part of z, and Im(z) is y, the imaginary part of z. Also $|z|$ is the *modulus* of z, defined as $|z| = \sqrt{x^2 + y^2}$.

2 Group Actions

The basic idea of a group action, as we saw in the introduction, is that elements of a group can provide information about how to "move around" (permute) the elements of a set, possibly keeping some elements fixed. The dihedral group elements give information about how to move around the vertices of a regular polygon. The group of 2×2 invertible matrices with entries in $\mathbb{R}$ describe a way to send vectors in $\mathbb{R}^2$ to each other. Notice, since we have restricted to invertible matrices, we now have a group acting on a set!

How do we formalize the idea that each element of the group permutes the set S? We define a bijective function $\varphi_g : S \to S$ for each $g \in G$ which describe how each group element moves around the elements of S. As with algebraic objects such as groups or rings, group actions have axioms. It is not enough that an element of the group merely shuffles elements of the set. There should be a nod to the group structure, otherwise it would just be a set acting bijectively on a set! For example, we should expect the identity element of the group to behave as the identity element always does, and leave all the elements of the set alone. Also, multiplication in the group should correspond to "multiplying" the functions φ_g defining the actions, except the natural way to "multiply" functions is to compose them. If we multiply two elements of the group together first and then consider what their product does to the set, that should be the same shuffling of the set as applying the second group element (numbered from left to right) to the set and then applying the first group element to the once shuffled set. In precise mathematical statements:

Definition 1. Let G be a group and S a set. A **group action** is a set of functions $\varphi_g : S \to S$, one function for each element $g \in G$, so that φ_{1_G} is the identity map, and $\varphi_{g_1 g_2} = \varphi_{g_1} \circ \varphi_{g_2}$.

We have seen two examples of group actions in the introduction. How do these examples satisfy the definition?

Example 1. When S is the set of vertices of a regular n-gon and $G = D_n$, then for each $g \in G$ the map φ_g is defined by sending each vertex to the vertex it goes to under the symmetry g. Note φ_{1_G} fixes all vertices, and since the group operation of G is function composition, the second group action axiom also follows.

Example 2. The set of all 2×2 matrices with entries in $\mathbb{R}$ under the operation of matrix multiplication is not a group as not all matrices have multiplicative inverses, so we must restrict ourselves to matrices with nonzero determinant. With that restriction, and with S the set of vectors in $\mathbb{R}^2$, the discussion in the introduction naturally defines the group action. For each invertible 2×2 matrix A, the map $\varphi_A : S \to S$ is defined by multiplying the matrix A by each vector in $\mathbb{R}^2$. It is clear that the identity element of the group (the matrix I) fixes all vectors. The second group action axiom is satisfied if $(A_1 A_2)(v) = A_1(A_2(v))$ which is true simply by the way we define matrix multiplication.

Example 3. A group can act on itself. Here the set S is equal to the group G. One way to describe an action of G on itself is by defining $\varphi_g : S \to S$ as $\varphi_g(h) = ghg^{-1}$ (*conjugation* by g). Pause here to check that the group action axioms are satisfied. Can you also come up with a different action of a group on itself?

You may be worried that in the definition I did not specify that φ_g must be a bijection. Fear not! The map φ_g has an inverse for each $g \in G$: the function $\varphi_{g^{-1}}$. Use the definition to show that this is, indeed, the inverse. You might see other definitions if you search for the phrase "group action" on the internet or in algebra books. For example, given a group G and a set S, a group action is a map from $G \times S$ to S with the image of the value (g, s) called $g \cdot s$, and with two properties: (1) $1_G \cdot s = s$ for all $s \in S$ and (2) $(gh) \cdot s = g \cdot (h \cdot s)$ for all $g, h \in G$ and $s \in S$. Although the group action axioms using these other definitions may look different to you on first inspection, it is a nice exercise to prove they are actually equivalent to the definition I give above.

Now that we have a formal definition and a couple of examples under our belt, there are a few sets we need to define which will be important later in the paper, and which allow us to more richly describe each particular group action. Elements of the set S that are fixed (unmoved) by an element of the group play an important role in understanding group actions. If we pick one element of the set S, call it s, then the set of elements of G which fix s is the **stabilizer of** s. In math symbols we write $G_s = \{g \in G \mid \varphi_g(s) = s\}$. To get comfortable with this definition, here are some questions to consider. Where does the stabilizer live, i.e. what is it a subset of? Can the stabilizer ever be the empty set? Does the stabilizer have any additional structure to it, e.g. is it a subgroup?

Example 4. For the action of D_n on vertices of a regular n-gon, reflection is in the stabilizer of the first vertex since this symmetry fixes that vertex, and if n is even it is also in the stabilizer of the $\left(\frac{n}{2}+1\right)$st vertex, since the line we reflect across also goes through this vertex. For the action of linear transformations on $\mathbb{R}^2$, the stabilizer of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ includes the matrix $M = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$. For an abelian group acting on itself by conjugation, the stabilizer of every element is the whole group, since $ghg^{-1} = gg^{-1}h = h$ for all $g, h \in G$.

Connected to the stabilizer are two other adjectives used to describe a group action.

Definition 2. A group action is **faithful** if for all nonidentity elements $g \in G$ there is an $s \in S$ so that $\varphi_g(s) \neq s$. This definition means that no nonidentity element of G can be in all the stabilizers, or that the intersection of all stabilizers of elements in S contains only the identity element. Here is a good place to pause and make sure you believe the first and second sentence are really saying the same thing mathematically.

Definition 3. A group action is **free** if whenever there is some $s \in S$ so that $\varphi_g(s) = s$, then g is the identity. Any such g would be in the stabilizer of s and so this definition says that the stabilizers of all elements of S are trivial (only contain the identity element). A free action is also faithful, although the converse is not generally true.

Another important set is the **orbit** of an $s \in S$, denoted $\mathcal{O}_s$. This is the set of elements of S which can be "reached" from s under the action. Alternatively $\mathcal{O}_s$ is the image of s under all of the functions φ_g. In math symbols we write $\mathcal{O}_s = \{t \in S \mid \varphi_g(s) = t \text{ for some } g \in G\}$. For orbits, consider the same questions I asked you about the stabilizer.

Example 5. When D_n acts on the set of vertices of a regular n-gon there is only one orbit, as repeated applications of rotation will move any vertex to any other vertex. In the case of matrices acting on vectors in $\mathbb{R}^2$, there are two orbits. One contains just the zero vector, and all the other vectors are in the other orbit. To show this, prove that given any two nonzero vectors $\vec{v_1}$ and $\vec{v_2}$ in $\mathbb{R}^2$, there is an invertible matrix M so that $M\vec{v_1} = \vec{v_2}$. For a group acting on itself by conjugation, the orbits are given the name *conjugacy classes*.

When a group G acts on a set S, we can create the set of all orbits of this action. It turns out that if we take $r, s \in S$ then $\mathcal{O}_r$ and $\mathcal{O}_s$ either intersect trivially or are identical (in math terminology we say that the orbits *partition* the set S). One way to prove this is to show that the relation defined as

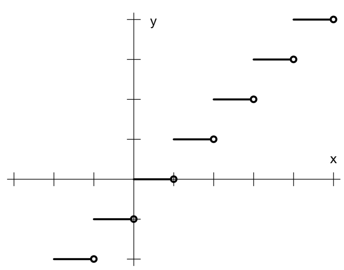
$$r \sim s \text{ if and only if there is a } g \in G \text{ so that } \varphi_g(r) = s$$

is an equivalence relation, because an equivalence relation partitions a set. Proving the assertions above would be a good refresher on equivalence relations.

The terminology of group actions is naturally descriptive. The *stabilizer* is the set of group elements which keep an element of the set stable, or unmoving! Think of *orbit* like in the phrase "my social orbit" in reference to your extended circle of friends and social acquaintances.

3 Riemann Surfaces

One of the key concepts in a calculus of one variable course is that "nice" functions (usually meaning functions that are differentiable everywhere) are those that look like lines close up. Lines are basic objects we all understand well, so if we can approximate a complicated function with a bunch of lines, we reduce the complexity of the function, at least locally. We say *locally* to mean in a small neighborhood of a point. But, we also want those different lines to interact well with each other. The step function can be approximated by lines, but those discontinuous points (the jumps) make the step function a less desirable function, from a calculus point of view. We cannot, for instance, define a derivative everywhere on this function. Or consider the absolute value function. It can be approximated by two lines ($y = x$ if $x \geq 0$ and $y = -x$ if $x \leq 0$), but the slope of those two lines do not coincide on their intersection (at $x = 0$) and so we cannot define a derivative at the point $x = 0$. No matter how tiny the neighborhood around the point $x = 0$ is, the absolute value in that neighborhood will never look like a line, it will always be V-shaped. See Figure 2 if you forget what these functions look like.



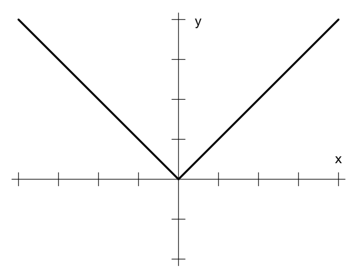


Figure 2: The step function $f(x) = \lfloor x \rfloor$ (left) and the absolute value function $f(x) = |x|$ (right).

A similar concept exists in higher dimensions. Instead of functions in one variable, we consider *topological spaces*. If you have not seen any topology, the definition of topological spaces is in Section 8. Following convention we sometimes call topological spaces simply "spaces". We will talk of *points* on spaces, and a *neighborhood* around the point, by which we mean an open set containing the point. Informally the topological spaces called *Riemann surfaces* we discuss in this paper are like surfaces in $\mathbb{R}^3$, specifically those surfaces which locally resemble a plane.

While most of the analysis you have seen so far likely featured real numbers, there are several advantages to analysis over the complex numbers. In real analysis, there are functions that are differentiable but whose derivatives are not differentiable. A way to construct such a function is to start with your favorite continuous, but not differentiable, function. Mine is $f(x) = |x|$. Then the Fundamental Theorem of Calculus produces a function which is differentiable, but whose derivative is your favorite continuous-but-not-differentiable function. In my case, one example would be the function $F(x) = \int_{-1}^{t} |t|\, dt$ on the interval $[-1, 1]$. In real analysis, there are also functions that do not equal their Taylor series. The function $g(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is an example. It takes some work with limits, but you can show all of the derivatives of $g(x)$ at $x = 0$ evaluate to 0. So the Taylor series expansion at 0 is just the zero function, which is certainly not equal to $g(x)$ anywhere except at $x = 0$.

Neither of these scenarios occur in complex analysis: if a function is differentiable, it is infinitely differentiable, and the Taylor series of a differentiable complex function will equal the function inside the radius of convergence (of the Taylor series). Therefore, we want to study complicated topological spaces which look like the complex plane locally.

Here is how we formalize this idea on such spaces. Around any point on the space, we define a map from a neighborhood of that point to some open subset of the complex plane. If we want to apply the results of complex analysis to the space, we use the map to move from the space to the complex plane, apply the analysis results there, and then use the inverse of the map to return to the space. This plays an analogous role to that of describing complicated real single variable functions locally by lines. In that case, think of the map as sending a neighborhood near a point p to a line (the *tangent line*, in fact).

There are a few technicalities to work out. We need the map to have an inverse, so the map must be a bijection. Additionally, the map has to allow any complex analysis we do in the open subset of $\mathbb{C}$ to transfer back to the space in a way that "preserves the topological structure". In your algebra course you saw the idea of an *isomorphism*, a bijective map from one group (or ring or field) to another group (or ring or field) which preserves the structures of the group (or ring or ... you get the idea!). For groups, to "preserve the structure" means we get the exact same result whether we multiply elements of the preimage and then apply the map to the product, or apply the map to elements of the preimage and then multiply the results in the image. In symbols, if f is the map, and $a, b \in G$, then $f(ab) = f(a)f(b)$. This (together with the bijective condition) is enough to ensure that isomorphic groups have the same cardinality, and that they share other important characteristics like orders of their elements or abelian-ness. As another example of preserving structure, in linear algebra, you learned that linear transformations corresponding to orthogonal matrices preserve distance.

In topology, the structure-preserving bijective maps are called *homeomorphisms*. To "preserve structure" here means preserving the topological structure, so the map must send open sets (and only open sets) to open sets. Look for a formal definition in Section 8. Thus our definition for the maps which give the local approximations of the surface by open subsets of the complex plane is:

Definition 4. Given a topological space X, a point p on X, a neighborhood U containing p, and an open subset $W \subseteq \mathbb{C}$, then a homeomorphism $\phi_U : U \to W \subseteq \mathbb{C}$ is called a **complex chart**.

One other technicality is if two neighborhoods U and V around different points intersect, the two complex charts ϕ_U and ϕ_V should behave similarly on that intersection. We would get into trouble in later arguments if we had to worry about which chart we were picking, and if those choices affected the outcome. Remember, our goal is to be able to do analysis on the open subsets of $\mathbb{C}$, and then lift that back up to the surface X via the inverse of the complex chart. To make two charts compatible with one another, doing the lift back to X and then applying a different chart to get to a (possibly different) open subset of $\mathbb{C}$ should preserve analysis properties. We specify that the map $\phi_V \circ \phi_U^{-1}$ should be a *holomorphic map*, one that is complex differentiable at each point and, as such, preserves the analysis properties. This leads to the last two definitions we need before we can properly define a Riemann surface.

Definition 5. Two complex charts ϕ_U and ϕ_V are called **pairwise compatible** if $\phi_V \circ \phi_U^{-1}$ is holomorphic. A set of pairwise compatible complex charts so that each point in X is in at least one of the domains of the charts is called a **complex atlas** for X.

Riemann surfaces are topological spaces equipped with a complex atlas, and with a couple of other small restrictions which ensure the spaces behave "reasonably".

Definition 6. A topological space X which is Hausdorff, connected, and second countable, and which has a complex atlas defined on it is called a **Riemann surface**.

You can find the definitions for *Hausdorff*, *connected*, and *second countable* in the definition section. As you are first grappling with the concept of a Riemann surface, do not worry too much about these additional definitions. Most "nice" topological spaces, and all the examples in this paper, satisfy them.

Example 6. Consider $\mathbb{R}^2$, and fix any element $(x, y) \in \mathbb{R}^2$. Then the map $\phi : \mathbb{R}^2 \to \mathbb{C}$ sending $(x, y) \to x + iy$ is a homeomorphism, and a complex chart, and so $\mathbb{R}^2$ is a Riemann surface with the atlas $\{\phi\}$.

Example 7. Any open subset of the complex plane is a Riemann surface, by defining one complex chart from the open subset projecting onto itself. For example, the *open unit disk* $D = \{z \in \mathbb{C} \mid |z| < 1\}$ is a Riemann surface with $\phi : D \to D \subseteq \mathbb{C}$ defined as $\phi(z) = z$, and atlas $\{\phi\}$. A similar argument follows for the *upper half plane* $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$.

Example 8. Sometimes a non-example can be just as enlightening. Consider the chart from Example 6, but restrict it to the open set of elements $(x, y) \in \mathbb{R}^2$ so that $x > -1$. Call that chart ϕ_{U_1}, and consider a second chart ϕ_{U_2} defined as $\phi_{U_2}((x, y)) = x - iy$ on the open set of elements $(x, y) \in \mathbb{R}^2$ so that $x < 1$. In order for $\{\phi_{U_1}, \phi_{U_2}\}$ to be an atlas on $\mathbb{R}^2$, the two charts must be compatible on the intersection of the open sets, i.e. the open strip between $x = -1$ and $x = 1$. But on this intersection $\phi_{U_2} \circ \phi_{U_1}^{-1}$ sends $x + iy$ to $x - iy$, which is not a holomorphic map. So $\{\phi_{U_1}, \phi_{U_2}\}$ is not an atlas on $\mathbb{R}^2$.

How do we know that the map sending $x + iy$ to $x - iy$ is not holomorphic? Complex functions can be described as having a real part and an imaginary part which are themselves real functions. There is a set of differential equations relating these real functions, called *the Cauchy-Riemann equations*, that are satisfied precisely when the corresponding complex function is holomorphic. In effect, the Cauchy-Riemann equations give a real analysis condition for when a complex function is differentiable. For our example, $\phi_{U_2} \circ \phi_{U_1}^{-1}$ can be written as $u(x, y) + v(x, y)i$ where $u(x, y)$ is a real function in two variables sending (x, y) to x, while $v(x, y)$ sends (x, y) to $-y$. One of the Cauchy-Riemann equations is $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, but in our case, this equation is not satisfied as $\frac{\partial u}{\partial x} = 1$ but $\frac{\partial v}{\partial y} = -1$. And so $\phi_{U_2} \circ \phi_{U_1}^{-1}$ is not holomorphic.

Before giving another example, we pause here to discuss what it means for two Riemann surfaces to be equivalent.

Definition 7. Given Riemann surfaces X and Y, a map $f : X \to Y$ is **holomorphic** at a point p if there is a chart ϕ_U of X with $p \in U$ and a chart ϕ_V of Y with $f(p)$ in V so that $\phi_U \circ f \circ \phi_V^{-1}$ is holomorphic. If such charts exist for all $p \in X$ then f is a **holomorphic map**.

This definition is a good example of the motivation for defining charts as we do. There is no way to define a derivative on an arbitrary surface. But, because Riemann surfaces look like complex planes locally, we can use the definition of holomorphic in $\mathbb{C}$ and translate it back to X using charts, giving us a definition of a holomorphic map on X. Notice also that pairwise compatibility of charts is important in these definitions so that we do not need to worry about the choice of ϕ_U and ϕ_V. If we choose different charts for points in X, the holomorphicity still carries through.

Example 9. We can define a holomorphic map between the two Riemann surfaces in Example 7. You should check that the map $f : \mathbb{H} \to D$ with $f(z) = \frac{z-i}{z+i}$ actually sends elements in the upper half plane to elements in the open disk, and that f is a holomorphic map. There is also a holomorphic inverse map $f^{-1} : D \to \mathbb{H}$ defined as $f(z) = -i\frac{z+1}{z-1}$ and so $\mathbb{H}$ and D are *isomorphic* Riemann surfaces.

Another popular example of a Riemann surface, and one we will use later in the paper, is the *Riemann sphere*.

Example 10. Take the unit sphere in $\mathbb{R}^3$, defined as $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, call the points $(0, 0, \pm 1)$ the *north and south poles*, and consider the (x, y) plane which, as we just saw in Example 6, is homeomorphic to $\mathbb{C}$. To ease our conversation, consider the points (x, y) written as $x + iy \in \mathbb{C}$. Through a process called *stereographic projection*, we will define two charts from $S^2 - \{(0, 0, 1)\}$ and $S^2 - \{(0, 0, -1)\}$ to the complex plane (represented as the (x, y) plane).

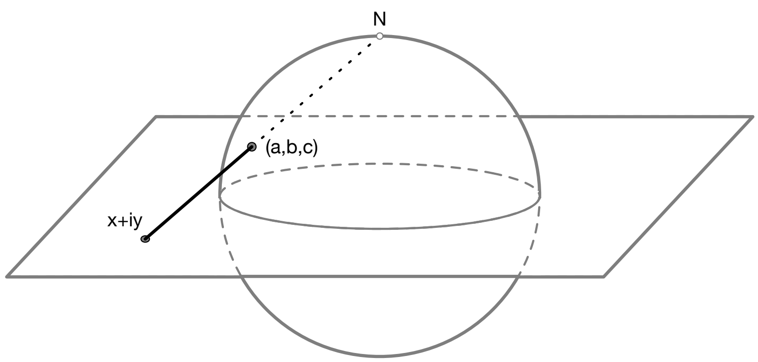


Figure 3: The line forming the bijection between points on S^2 minus the north pole and points on the complex plane.

Begin with the north pole (labeled as N in Figure 3). Any line through the north pole will hit the sphere in exactly one place and will hit the complex plane also in exactly one place. These lines, therefore, define a bijection from points on $S^2 - \{(0, 0, 1)\}$ to points on $\mathbb{C}$. In your multivariable calculus class, you learned how to define lines in $\mathbb{R}^3$, so the line going through $(0, 0, 1)$ and intersecting the sphere at a point $(a, b, c) \in S^2$ can be parametrized as $\ell(t) = (0, 0, 1) + (a, b, c - 1)t$. Where will this line intersect the (x, y) plane? When the z coordinate is 0, or, solving for t in ℓ, when $t = 1/(1 - c)$. Simply plug this value of t into $\ell(t)$ to get a map $\phi_N : S^2 \to \mathbb{C}$ so that $\phi_N(a, b, c) = \frac{a}{1-c} + \frac{b}{1-c}i$.

The map ϕ_N is a bijection. Its inverse map ϕ_N^{-1} is created in much the same way as we created ϕ_N. Take some $x + iy \in \mathbb{C}$ and form the line from the north pole to $x + iy$ (or really from its equivalent point in $\mathbb{R}^3$: $(x, y, 0)$), and determine where that line intersects the sphere S^2. You will get a formula $\phi_N^{-1}(x, y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$.

To make this map a chart, we do need to confirm that it is homeomorphic. How do we know ϕ_N and its inverse are homeomorphic (i.e. continuous)? We can say that both maps are compositions of known continuous functions (polynomials and rational functions) and thus are continuous.

Now, go through an almost identical argument yourself using the south pole to create a second complex chart $\phi_S : S^2 \to \mathbb{C}$ so that $\phi_S(a, b, c) = \frac{a}{1+c} - i\frac{b}{1+c}$. Before we can declare the set $\{\phi_N, \phi_S\}$ to be an atlas on S^2, we must ensure the two charts are pairwise compatible. The composition $\phi_S \circ \phi_N^{-1} : \mathbb{C} \to \mathbb{C}$ sends z to $1/z$, which is a well known holomorphic map. Once more I leave the thrilling calculations to you, dear reader.

The Riemann sphere is often called $\mathbb{C} \cup \{\infty\}$, since S^2 minus the north pole is homeomorphic to $\mathbb{C}$, and we can think of N as the limit point of any unbounded sequence in $\mathbb{C}$, hence a "point at infinity". We will discuss the process of adding a point in the next section, as it connects to a particularly nice form that all of the Riemann surfaces in my research have. Then in Section 4.2 we return to examples of Riemann surfaces by considering a special family I use in my research.

3.1 Compactness

Sets which are closed and bounded in $\mathbb{R}$ have nice properties. As just one example, the Extreme Value theorem says that on a closed and bounded subset of $\mathbb{R}$, any continuous function attains an absolute maximum and minimum. We would like to extend the ideas of closed and bounded sets beyond $\mathbb{R}$. This is where a property called *compactness* comes in. In your introductory analysis course you may have seen a definition for compactness involving open covers. The topological spaces we care about in this paper are special cases of objects called *metric spaces*, and in metric spaces this definition of compactness is equivalent to the following one, which will be easier to work with in our setting.

Definition 8. Topological spaces in this paper are **compact** if every sequence has a convergent subsequence.

Is the real line compact? (If compactness is a generalization of closed and bounded sets then the answer had better be no!) The sequence $(1, -1, 1, -1, 1, -1, \ldots)$ in $\mathbb{R}$ does not converge, but does have a convergent subsequence consisting of every even or every odd entry. The sequence $\left(1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, \ldots\right)$ in $\mathbb{R}$ has a subsequence that converges to 0. But not all sequences have convergent subsequences. Take the sequence of integers $(1, 2, 3, 4, \ldots)$. This sequence most definitely does not have a convergent subsequence as the limit of every subsequence will still be ∞. So $\mathbb{R}$ is not a compact Riemann surface.

The failure of compactness in $\mathbb{R}$ boils down to a problem with unbounded sequences as, by a result you probably saw in your real analysis course called the Bolzano-Weierstrass theorem, any bounded sequence $\mathbb{R}$ will have a convergent subsequence. Unbounded sequences in $\mathbb{R}$ all have a subsequence that heads to $\pm\infty$ and that subsequence is the crux of the problem. If we could just add a point called infinity to $\mathbb{R}$, and define that point to be the limit of the unbounded sequences, then we would have a compact space. Geometrically, imagine identifying $\pm\infty$ as one new point and, by adding that one point, producing a circle. We have thus created a compact space $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ simply by adding one point to $\mathbb{R}$. The process of turning a surface which is not compact into a compact one by adding points has the fun name of "compactification"!

The complex plane is also not compact for similar reasons: the sequence of integers $(1, 2, 3, 4, \ldots)$ does not have any convergent subsequences. We can fix this (create a compactification of $\mathbb{C}$) by adding a point at infinity. In Example 10 we described just such an object, called the *Riemann sphere*.

Since intuitively the idea of compactness should be connoted with the ideas of closed and bounded, compact spaces should not go off to infinity (like the upper half plane $\mathbb{H}$ from Example 7) but instead should fold in on themselves, and they should not be open (like the unit disk D also in Example 7). When you think of compact Riemann surfaces, you can have in your mind hollow shapes in $\mathbb{R}^3$ like a sphere or a torus (holed donut) or a many holed torus.

3.2 Curves

The family of compact Riemann surfaces I work with are *curves*. You are well acquainted with curves, at least over $\mathbb{R}$. For example, the unit circle, defined as the set of solutions in $\mathbb{R}^2$ to the equation $x^2 + y^2 = 1$, is a curve. Notice we could also define it as solutions to $f(x, y) = 0$ where $f(x, y) = x^2 + y^2 - 1$. Similarly for curves over $\mathbb{C}$, take some polynomial $f(z, w) \in \mathbb{C}[z, w]$, the polynomial ring in two variables with coefficients in $\mathbb{C}$, and assume the degree of f is greater than 0 to avoid some pathological cases. Let

$$X = \{(z, w) \in \mathbb{C}^2 \mid f(z, w) = 0\}. \qquad (1)$$

This set X is a (plane) **curve**.

Near the beginning of Section 4, we describe several analytic advantages to studying the complex plane. In addition, the complex numbers are *algebraically closed*, which means any polynomial with coefficients in $\mathbb{C}$ has all its roots in $\mathbb{C}$. While the polynomial $x^2 + 1$ has no roots in $\mathbb{R}$, if we consider the same polynomial over $\mathbb{C}$, we see its two roots: $\pm i$. Therefore, we study curves that are defined by polynomials with coefficients in $\mathbb{C}$ both to align analytically with Riemann surfaces, and also so we do not need to worry about whether all roots of the polynomials exist in the field we are working in.

To define a Riemann surface structure on the curve, we must add some extra conditions. First, we want the curve to be connected, one of the conditions in our definition of Riemann surfaces. One way to guarantee connectedness is to insist that the polynomial $f(z, w)$ above be irreducible. The other condition is that we want the curve to be *smooth*, meaning it has a defined tangent at all points. From the definition of the tangent to a curve, this means the partial derivatives of f exist for each point on the curve, and are never simultaneously zero. For the rest of the paper, if I use the word "curve", assume I mean a smooth, connected curve.

How do we define complex charts for curves? For each point $p \in X$, we want an open neighborhood U containing p to exist so that the projection map $\pi_U : U \to W \subseteq \mathbb{C}$ sending (z, w) to z is a homeomorphism, and thus a complex chart. Projection is continuous since if we take an open set around a point z_0, its preimage under the projection will also be an open set.

When will π_U be injective? The answer to this question requires the Implicit Function theorem. This theorem is difficult to prove, but you have certainly used versions of it. Take our original example of a curve, the unit circle defined as $f(x, y) = x^2 + y^2 - 1 = 0$ which is not a function in $\mathbb{R}$. However, we can define subintervals on which $f(x, y) = 0$ can be described as a function. For instance, the top half of the circle is equivalent to $g(x) = \sqrt{1 - x^2}$ which is a function. Or for the bottom half, we can use $g(x) = -\sqrt{1 - x^2}$. The only points in $\mathbb{R}^2$ for which we cannot use either $g(x)$ defined above are the points $(\pm 1, 0)$. No matter what neighborhood you take around either of these two points, one x value will always correspond to two y values, and so there is no way to define a function $g(x)$ which represents the circle in a neighborhood around $(\pm 1, 0)$. However, the same is not true if we switch the roles of x and y. Near the points $(\pm 1, 0)$ each y value corresponds to exactly one x value and we can instead use the functions $g(y) = \pm\sqrt{1 - y^2}$ to locally define the circle. This is the idea of the Implicit Function theorem. For curves in $\mathbb{C}^2$ given by an equation $f(z, w) = 0$ and with special conditions on the partial derivatives, we are able to define holomorphic functions that are equal to the defining equation of the curve in small neighborhoods, although these functions may not all be defined with the same variable.

Back to the quest for complex charts in X. Take a curve X defined as in (1) and any point $p = (z_0, w_0)$ on it. Since the curve is smooth, we know that at least one partial derivative is nonzero at the point p. Without loss of generality, say it is the derivative with respect to w_0. Since that partial derivative is nonzero, we can apply the Implicit Function theorem, to conclude there is a neighborhood $U \subseteq X$ containing the point p so that every point in U is of the form $(z, g(z))$ for some function $g(z) \in \mathbb{C}[z]$. The key here is each z corresponds to just one w value in U which ensures that the map π_U has an inverse sending z to $(z, g(z))$, and so π_U is a bijection on U. We already said π_U is continuous, and the Implicit Function theorem tells us that its inverse is continuous, hence π_U is a homeomorphism. If the partial with respect to w_0 had been 0, then we would define the projection $\pi_U : U \to W \subseteq \mathbb{C}$ sending (z, w) to w and with inverse sending w to $(g(w), w)$ for a function $g(w)$ which exists by the Implicit Function theorem.

Now we have charts, the π_U, but are they pairwise compatible? If π_U and π_V are two charts so that $U \cap V \neq \emptyset$ and both are defined as projection on the same variable, then $\pi_V \circ \pi_U^{-1}$ is the identity map which is certainly a holomorphic map. If, say, π_U is projection onto the first variable, and π_V is projection onto the second variable, then $\pi_V \circ \pi_U^{-1}(z) = \pi_V((z, g(z))) = g(z)$ i.e. $\pi_V \circ \pi_U^{-1}$ is equal to the function $g(z)$ which is holomorphic by the Implicit Function theorem. Thus the curve X is a Riemann surface with complex charts ϕ_U.

I have hidden until now one flaw with our discussion about curves as Riemann surfaces. The curves we define above are called *affine curves* and the corresponding Riemann surface is not compact. In order to make the Riemann surface compact, we instead have to work with objects called *projective curves*. These are curves defined over *projective space* which, similar to our conversation about compactness in Section 4.1, is a space formed by adding a point at infinity. This is a tale for another time, but if you would like to read more about projective space, I recommend [3, Chapter 8].

It is also true that every compact Riemann surface is a projective curve, but that is not easy to prove! An overview of this result, including several references on where to find more details, is in Theorem 4.16.1 in [12]. The idea that certain analytic objects (compact Riemann surfaces in this case) are equivalent to certain algebraic objects (projective curves in this case) is a very deep connection between algebraic and analytic geometry, with the catchy name of GAGA, the acronym for the French title of the main paper connecting these ideas: Serre's "Géometrie Algébrique et Géométrie Analytique" [23] . The beauty of this connecting theory is that now we can bring both algebraic and analytic tools to bear on problems about these objects, and this opens up many more areas for exploration and understanding. I doubt GAGA has anything to do with the musician Lady Gaga, although I like to imagine a world where she chose her name in part because of this area of mathematics!

4 Groups Acting on Riemann Surfaces

Suppose we have a group G acting on a Riemann surface X. Recall from Section 3 that the orbits of a group action partition the set. For a given group G acting on X, this means the points on X may be collected into compartments, based on what orbit they are in. We call the set of orbits (the compartments) X/G. Geometrically, what is this set of orbits? Instead of a group action, first think more generally of equivalence relations on the points on X, as you may recall from Section 3 that a group action produces an equivalence relation.

As an example, take the closed disk $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and define an equivalence relation as z_1 and z_2 in D are equivalent if they are both on the boundary, or if $z_1 = z_2$. The equivalence relation partitions D into the boundary (as one compartment) and each point on the interior as its own compartment. What happens geometrically if we identify all the points on the boundary, i.e. what does the set of partitions look like? Visually, imagine shifting the disk into three dimensions by pulling up on its boundary, creating a bowl shape. Then push together the rim of the bowl to one point, forming a sphere. This tells us the set of partitions for this equivalence relation is (homeomorphic to) the sphere S^2.

The same idea transfers over to the special case of a group acting on a Riemann surface. Geometrically, we glue together into one point all the points in X which are in the same orbit (not so unlike the idea of *tessering* [14]), and do that for each orbit, which suggests we can put a topological structure on X/G too. The set of orbits X/G has the *quotient topology* inherited from the topology of X via the natural map $\pi : X \to X/G$ sending any point p to its orbit $\mathcal{O}_p$. The precise definition of quotient topology is found in the definition section. On a first reading, it is fine to take on faith that X/G is a topological space.

Here we are interested in the fact that X/G is also a Riemann surface. To make things a bit nicer, we assume the action of G on X is *faithful*, i.e. the intersection of all the stabilizers is trivial (see Section 3). This condition is not very restrictive, as it turns out that a non-faithful action of G on X can be represented by a faithful action (for the experts: quotient G out by the kernel of the action). We also assume, to preserve analytic structure, that each of the $\varphi_g : X \to X$ defining the group action is a *holomorphic map on* X (we defined a holomorphic map between Riemann surfaces in Section 4).

What are charts for X/G? For each point in X/G (an orbit $\mathcal{O}_p$ where $p \in X$), we need to find an appropriate neighborhood $U \subseteq X/G$ around $\mathcal{O}_p$ and a homeomorphic map ψ_U from U to an open subset of $\mathbb{C}$ so that the different ψ_U are pairwise compatible. Where could we get such a map ψ_U? Your instincts may be screaming at you to consider the charts we already have from the atlas of X, call them ϕ_V where V is a neighborhood of a point in X. We also have at our disposal the projection $\pi : X \to X/G$. Maybe we can use some sort of inverse of π composed with one of the charts of X to get a map $\phi_V \circ \pi^{-1} : U \to W \subseteq \mathbb{C}$?

The issue we encounter is that to properly define an inverse of π, we need a one-to-one map, at least when restricted to some open subset of X. For an open subset $V \subseteq X$, call the restricted map $\pi|_V : V \to X/G$. If $\pi|_V$ were one-to-one, this would mean whenever $\pi|_V(p_1) = \pi|_V(p_2)$ then $p_1 = p_2$. And what does it mean if $\pi|_V(p_1) = \pi|_V(p_2)$? It means p_1 and p_2 are in the same orbit under the action of G, or there is some $g \in G$ so that $\varphi_g(p_1) = p_2$. Thus $\pi|_V$ will be one-to-one if V contains no distinct elements from the same orbit.

To execute our plan of using charts of X together with π to define charts on X/G, we therefore need to find, for each point in X, a neighborhood V with not more than one element from each orbit in it. It turns out that when a group G acts on a Riemann surface X faithfully and holomorphically we can always find such V for any p which has trivial stabilizer. Construction of these neighborhoods is a bit technical, but the interested reader can find all the details in [16, Proposition III.3.3]. The Hausdorff property of Riemann surfaces plays a role here, as we can separate each of the distinct points in the orbit into different neighborhoods.

Equipped with this one-to-one map $\pi|_V$, we create the charts in X/G. Start with some point in X/G, call it $\mathcal{O}_p$. Form the neighborhood V in X of p as discussed above, so that the quotient map π restricted to V is a homeomorphism onto a neighborhood U of $\mathcal{O}_p$ in X/G. Now, because X is a Riemann surface, there is a chart of X defined as $\phi_V : V \to W \subseteq \mathbb{C}$ (we may need to shrink V to apply the chart). And so $\psi_U = \phi_V \circ \pi|_V^{-1} : U \to W$ is a chart for the neighborhood U of $\mathcal{O}_p$. It is homeomorphic since it is a composition of homeomorphic functions. Why are these charts pairwise compatible? Write out $\psi_{U_2} \circ \psi_{U_1}^{-1}$ as a composition of ϕ_{V_i} and $\pi|_{V_i}$, and observe that the projection maps cancel out, leaving only $\phi_{V_2} \circ \phi_{V_1}$ which is holomorphic, since the ϕ_{V_i} are pairwise compatible.

The whole argument above does not quite work for points p with nontrivial stabilizers, but the general idea is still the same. Again, see [16, Proposition III.3.3] for the case of nontrivial stabilizers. Thus if a group G acts faithfully and holomorphically on a Riemann surface X, the set of orbits X/G will also be a Riemann surface.

One final point on the way out of this subsection. The image via a continuous function of a compact space is again a compact space and so X/G will be compact whenever X is because π is continuous.

4.1 Automorphisms

In the previous section, we talked abstractly about an arbitrary group acting on a Riemann surface. But what sort of groups act on Riemann surfaces? Geometrically, the action will be a shuffling of the points in X, so how can a group represent shuffling of points? Consider curves, which we saw are Riemann surfaces in Section 4.2. Suppose we have a curve X defined by the equation $f(x, y) = y^2 - x^6 - x^3 - 1 = 0$. If there is one solution (x_0, y_0) to the equation $y^2 = x^6 + x^3 + 1$, pause and think about what other solutions must then exist in $\mathbb{C}$.

Perhaps you came up with the solution $(x_0, -y_0)$. With a little more work, you may have found the solution $(\omega_3 x_0, -y_0)$ where ω_3 is a cube root of unity, that is $\omega_3 = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$, or the solution $\left(\frac{1}{x_0}, \frac{y_0}{x_0^3}\right)$ as long as $x_0 \neq 0$. These additional solutions lead to two new maps from X to X. Call the map r which sends any point (x_0, y_0) to $(\omega_3 x_0, -y_0)$, and call the map s which sends each point (x_0, y_0) to $\left(\frac{1}{x_0}, \frac{y_0}{x_0^3}\right)$, except for the point $(0, 1)$ which is fixed by s. These two maps define ways to permute the points on this curve. If we want a group action, we simply need to create a group generated by these two elements, under the operation of composition. For instance, $r \circ r = r^2 : X \to X$ sends (x_0, y_0) to $(\omega_3^2 x_0, y_0)$, and $r^3 : X \to X$ sends (x_0, y_0) to $(x_0, -y_0)$ (hey, that looks familiar!). Figure out higher powers of r and what the inverse of r is. Now do the same thing with s. Play around to see what $r \circ s$ and $s \circ r^{-1}$ are. Can you identify the group generated by r and s? (Hint: we probably asked you about $r \circ s$ and $s \circ r^{-1}$ for a reason!) This group is (isomorphic to) the dihedral group on 6 objects, or D_6. It turns out that this is the full automorphism group of this curve, but that is not easy to see.

Now we have an explicit example of how the dihedral group acts on the curve (Riemann surface) defined by the equation above. This example is a specific case of a much more general construction, as I am sure you can imagine finding similar maps given a different equation. Remember that when we studied X/G in the previous section, we restricted to actions defined by holomorphic maps. Well, the maps above are holomorphic maps with holomorphic inverses.

Definition 9. An **automorphism** of a Riemann surface X is a holomorphic map from X to itself which is also a bijection, and with an inverse that is also holomorphic. The set of automorphisms form a group under function composition called the **automorphism group**.

The automorphism group of a Riemann surface is one example of a group which can act on a Riemann surface. These groups are particularly relevant to my research, and I talk about them in Section 6. While it is not obvious, the automorphism groups of compact Riemann surfaces are all finite, and in fact bounded in size by an invariant of the surface called the *genus* of X. Topologically the genus measures how many holes a surface has, and for curves it roughly measures how big the degree of the defining polynomial is. The formal result is *Hurwitz's Theorem* which says that a Riemann surface X of genus g has an automorphism group G so that $|G| \leq 84(g-1)$ [16, III.3.9]

4.2 Riemann Existence Theorem

So far we have determined that if G is a group acting on a compact Riemann surface, then X/G is also a compact Riemann surface. We saw that automorphism groups of curves give examples of groups acting on Riemann surfaces, and for compact Riemann surfaces (remember, these are the same as projective curves) the groups are finite. Can we be more explicit about which finite groups are automorphism groups of some compact Riemann surface? Indeed we can, and the crux of this theory is the beautiful Riemann Existence theorem [8], a version of which we state here.

Each action of some group G on a compact Riemann surface X has a list of integers $[m_1, \ldots, m_r]$ associated to it which satisfy a formula called the Riemann-Hurwitz formula [2, Lemma 3.13]. This formula relates the size of G, the m_i, and the genus of X and X/G. Recall from the introduction that for any $g \in G$, the notation $o(g)$ represents the order of g.

Theorem 1. *Given a list of integers $[m_1, \ldots, m_r]$ satisfying the Riemann-Hurwitz formula, there is a finite group G and elements $g_1, \ldots, g_r \in G$ so that*

1. *for all i, $o(g_i) = m_i$,*
2. *the g_i together generate G, and*
3. *the product $g_1 g_2 \cdots g_r$ is the identity of G, if and only if G is the automorphism group of some compact Riemann surface X. In this case, the quotient X/G is isomorphic to the Riemann sphere.*

Example 11. For the alternating group A_6, the elements $g_1 = (1\,3)(4\,6)$, $g_2 = (1\,5\,6\,4)(2\,3)$, and $g_3 = (1\,2\,3\,4\,5)$ satisfy the Riemann Existence theorem for $m_1 = 2$, $m_2 = 4$, and $m_3 = 5$. The elements $g_1 = (1\,2\,3)(4\,5\,6)$, $g_2 = (1\,4\,2\,3)(5\,6)$, and $g_3 = (1\,2\,3\,4\,5)$ satisfy the Riemann Existence theorem for $m_1 = 3$, $m_2 = 4$, and $m_3 = 5$.

A topological proof of this theorem exists which is constructive (see [26, Theorem 4.27 and Theorem 4.32], although be warned that this is a very advanced book).

What is so powerful about the Riemann Existence theorem is that it connects topology, group theory, and complex geometry in the study of compact Riemann surfaces, giving us many powerful tools in each of these disciplines to attack problems with. In my humble opinion, this is where the most beautiful mathematics exists, along boundaries of different disciplines within mathematics. For my research, this theorem gives me a way to find explicit automorphism groups of compact Riemann surfaces by using group theory.

I am sure you are wondering why the conditions on the elements g_i in the theorem are what they are. The map $\pi : X \to X/G$ is more than just a quotient map, it is also a *covering*, branched at r places (r being the number of g_i in Theorem 1). If you have seen enough covering space theory, the g_i are generators of the monodromy group. As permutations they describe what happens to the preimages of the fixed base point in X/G when loops starting and ending at that base point and traveling once around the ith branch point are lifted to X. This idea will not be explored further here but if you are interested in learning more, a good graduate level reference for covering space theory and group actions related to them is Chapter 11 and 12 in [13]. For the specific situation discussed here, we refer you to [8] or [2] or [26].

You may also be wondering why the Riemann Existence theorem specifies that X/G is the Riemann sphere and if we could generalize to other Riemann surfaces. The answer is definitely yes, but things get more complicated if we consider other surfaces. For the experts in the room, if X/G has genus g_0, we have to add $2g_0$ generators of G and conditions on the products of commutators of these generators, corresponding to the fundamental group of a genus g_0 surface.

5 Contributions

My research path has led me to study objects called *Jacobian varieties*, and how they factor into smaller pieces. Much like the prime factorization trees we all remember from grade school (see Figure 4), many mathematical objects can be broken down into a product of simpler pieces. These pieces are often easier to study, and they give us additional information about the original object. In your linear algebra course you probably talked about decomposing matrices. For example, the QR decomposition is a way to write a real square matrix as the product of an orthogonal matrix and an upper triangular matrix. This decomposition lets us better understand how the linear transformation acts on vectors. You also saw an example in your introductory algebra class, in the form of *direct products*. If C_n represents the cyclic group of order n, then $C_6 \cong C_3 \times C_2$ and, perhaps more surprisingly, $D_6 \cong D_3 \times C_2$.

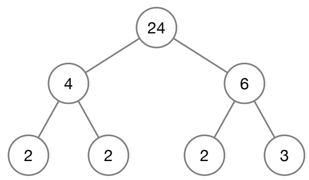


Figure 4: The factorization of the integer 24 into a product of prime integers.

While defining Jacobian varieties carefully is sadly beyond the scope of this paper (take a graduate level course in algebraic geometry and then read [15] or the relevant parts of [9]), I can give you a bit of information about them. First what are *varieties*? They are solutions to a system of polynomial equations. Curves (see Section 4.2) are a special example of varieties defined by one equation in two variables. Second, Jacobian varieties are *abelian varieties* which means they have a group structure on their points (not the group action we talked about, just an honest to goodness group formed naturally from the points). Finally, each Jacobian variety is connected in a canonical way to a curve, as the group structure of the Jacobian variety gives us information about the points on the corresponding curve.

Jacobian varieties can be viewed as a generalization of *elliptic curves*. If you have already been introduced to elliptic curves, you know that they have a beautiful and natural group structure on their points. If you have not seen elliptic curves before, or would like a refresher, I highly recommend [24]. The group structure of elliptic curves is very powerful, and the Jacobian variety group structure allows us to recover some of this power for curves which do not have a natural group structure on their points.

Fix your favorite (projective) curve X. Remember that projective curves are compact Riemann surfaces, so our discussions of group actions apply to the curve X. The symbol JX represents the Jacobian variety of this curve. My interest in factoring Jacobian varieties started with my Ph.D. thesis, where I explored how to leverage the group action of automorphisms of X to decompose JX into products of smaller abelian varieties [19] (here smaller means smaller topological dimension). I looked for examples where the Jacobian variety factored into the product of one elliptic curve, or $JX \sim E^g$, where E^g means the product of g copies of some elliptic curve E. Elliptic curves play a role similar to prime numbers in integer factorization: they are the "smallest" abelian varieties (they are all dimension 1) and so cannot be factored more. The $\sim$ symbol represents an *isogeny*, a map which is not quite an isomorphism, but close enough for most purposes, i.e. it preserves the structure we care about.

The decomposition technique requires a subject called representation theory (groups acting on vector spaces) as well as some more advanced topology. If you would like to read more about representation theory assuming only an undergraduate background, I recommend [11]. Side note: As an undergraduate I was not overly fond of linear algebra so I happily avoided taking a representation theory course in graduate school, convinced I would never, ever, ever need that material. The fact that I frequently use this topic in my research now is more evidence that the universe has a demented sense of humor, and that you never know where your mathematical life will take you!

In Section 5.1 we mentioned bounds for the size of the automorphism group of any curve of a fixed genus. There are families of groups which attain this bound. For example, the *projective special linear groups* $\mathrm{PSL}(2, q)$ are a family of groups defined by taking 2×2 matrices with coefficients in the finite field with q elements and with determinant 1 and forming the quotient group by the center, the subgroup $\{\pm I\}$. These groups are the automorphism groups of a family of curves, and of maximal possible order for the genera of the curves in that family. Since the representation theory of these groups is well known, finding the Jacobian variety decomposition of the corresponding curves is a tractable problem. Two undergraduates worked with me through a summer program at Grinnell College and determined this decomposition. [7]. Here is the key result they proved.

Theorem 2 (Theorem 10 [7]). *Fix a curve X with automorphism group PSL$(2, q)$ so that q is odd and $q > 27$, and which is of maximal possible order for the genus of X. Let A_x^y be the product of y copies of an abelian variety of dimension x while u, v, and w are values depending on $q \bmod 168$, and let $\phi(x)$ be the Euler phi function.*

When $q \equiv 1 \bmod 4$

$$JX \sim A^{q}_{\frac{q-u}{84}} \times A^{\frac{q+1}{2}}_{\frac{q-v}{84}} \times \prod_{\substack{d \mid \frac{q-1}{2} \\ d < \frac{q-5}{4}}} A^{q+1}_{\frac{\phi\left(\frac{q-1}{2d}\right)\cdot(q-w)}{168}} \times \prod_{\substack{d \mid \frac{q+1}{2} \\ d < \frac{q-1}{4}}} A^{q-1}_{\frac{\phi\left(\frac{q+1}{2d}\right)\cdot(q-w)}{168}}.$$

When $q \equiv -1 \bmod 4$

$$JX \sim A^{q}_{\frac{q-u}{84}} \times A^{\frac{q-1}{2}}_{\frac{q-v}{84}} \times \prod_{\substack{d \mid \frac{q-1}{2} \\ d < \frac{q-3}{4}}} A^{q+1}_{\frac{\phi\left(\frac{q-1}{2d}\right)\cdot(q-w)}{168}} \times \prod_{\substack{d \mid \frac{q+1}{2} \\ d < \frac{q-3}{4}}} A^{q-1}_{\frac{\phi\left(\frac{q+1}{2d}\right)\cdot(q-w)}{168}}.$$

In particular, these Jacobian varieties always factor into smaller pieces (they never behave like prime numbers). I reproduce this result to give you an idea of the kind of theorems we prove, not because I expect you to have a deep understanding of these Jacobian varieties simply by staring at their decompositions. If $q \leq 27$ and PSL$(2, q)$ is the automorphism group of a Riemann surface, then the only options for q are values in the set $\{7, 8, 13, 27\}$. In the first three cases, the Jacobian variety decomposes completely into a product of elliptic curves. If $q = 27$, we can still decompose the Jacobian variety. It just has a slightly different form which comes from the fact that the conjugacy class structures of PSL$(2, 27)$ is not the same as for larger q (for the experts: there are two conjugacy classes of elements of order 3 whereas for larger q there is only one).

Anita Rojas from Universidad de Chile and I have worked toward answering an open question from [5]. They ask if there are arbitrarily large curves so that the Jacobian variety of the curve is *completely decomposable*, meaning it factors into all elliptic curves (not necessarily all the same curve). By "arbitrarily large" we mean of arbitrarily large genus. They produce examples of such curves up to genus 1297, but there are many gaps in their list of examples ("gaps" here means missing *genera*, my favorite plural word in mathematics).

To find more examples, we created a new technique to decompose more Jacobian varieties [21]. If G is the automorphism group of some curve X, the quotient X/G (which, remember, is also a Riemann surface) may not have any automorphisms which prevents us from being able to decompose the Jacobian of X/G using old techniques. Our new method uses the decomposition of JX to induce a decomposition of the Jacobian of X/G, the thitherto indecomposable Jacobian variety. We applied this technique to many examples, using the Riemann Existence theorem to determine automorphism groups, and found many new genera with completely decomposable Jacobian varieties.

Theorem 3 (Theorem 1 [21])**.** *For every integer g in the following list, there is a curve of genus g with completely decomposable Jacobian variety found using a group acting on a curve.*

1–29, **30***,* *31,* **32***,* *33,* **34–36***,* *37,* **39***,* *40, 41,* **42***,* *43,* **44***,* *45,* **46***,* *47,* **48***,* *49, 50,* **51–52***,* *53,* **54***,* *55, 57,* **58***,* *61, 62,* **63***,* *64, 65,* **67***,* **69***,* **71***,* *72–73,* **79–81***,* *82,* **85***,* **89***,* **91***,* **93***,* **95***,* *97,* **103***,* **105–107***,* *109,* **118***,* *121,* **125***,* *129,* **142***,* *145,* **154***,* *161, 163,* **193***,* **199***,* **211***,* **213***,* *217,* **244***,* *257, 325, 433*

The numbers in bold are new examples, the other examples are all different from those in [5].

In another paper, I explore the factorization of Jacobian varieties of a family of curves called *hyperelliptic curves* [20]. These curves have nice defining equations of the form $y^2 = f(x)$ with $f(x) \in \mathbb{C}[x]$, and their automorphism groups are well known which makes proving more general results about the Jacobian factorizations possible.

When I introduced the Riemann Existence theorem, you may have asked yourself some questions such as, "Which groups satisfy it? Which elements of those groups satisfy it?" An algorithm which can determine groups and elements satisfying the Riemann Existence theorem for a fixed genus exists [2], and complete lists have been determined through genus 48. This data is very useful not only for my research (as you can see above), but also for other research mathematicians as well. Enter the *L-functions and Modular Forms Database* (LMFDB), a huge collaborative project among over 80 mathematicians to centralize mathematical data, with a particular goal of making it accessible to any researcher, regardless of their prior computing experience. I have added the data from [2], as well as several other key pieces of data here: `http://www.lmfdb.org/HigherGenus/C/Aut/`. Adding additional data is an ongoing project, as there is a lot of interesting information about group actions on compact Riemann surfaces which may be included.

6 Future Work

To intentionally misquote a proverb, "A mathematician's work is never done." Anita Rojas and I have produced more evidence toward answering the question in [5] about whether there are arbitrarily large curves with completely decomposable Jacobians, but the question has not yet been answered. It is likely that in order to make progress on the general question we will need to move beyond group actions to find other ways to factor Jacobian varieties.

Similar to the family of groups $\mathrm{PSL}(2, q)$, the alternating groups A_n are also the automorphism groups of many curves with maximal possible order for their genus. The representation theory of these groups is more complicated than for $\mathrm{PSL}(2, q)$, but students working with me have made some small progress in understanding these decompositions.

Due to the nature of the decomposition technique for Jacobian varieties, we often do not know which abelian varieties the factors are. In fact we rarely know equations defining even the original curves. Consider the statement of the Riemann Existence theorem. It says that if we find group elements with specific properties, then there is some curve which has that group as an automorphism group, but it does not give us a way to determine the equation of the corresponding curve. If we could find ways to determine which curves correspond to which automorphism group in higher genus, there are many interesting questions we could answer about the factors. Buzzwords for the experts: Do they have complex multiplication? What sort of torsion or rank do the elliptic curve factors have?

As I mentioned, there are a number of interesting questions related to Riemann's Existence theorem, and more generally to automorphism groups of curves. Right now, I am thinking about which lists of m_i produce examples for a given group. There are a number of combinatorial and group theoretic results that go into studying this question. A good reference to give you a sense of recent progress in this general area is a collection of research level papers [10].

7 Definitions

This section contains formal definitions to tangential ideas discussed in this paper, on topics you may have not yet seen in your undergraduate education. The definitions are given roughly in the order they appear in the paper, starting in Section 4. Most are in the area of topology or complex analysis. A standard reference for undergraduate topology material is [17]. I learned from [1] and remember enjoying the book as a student. Just about any undergraduate complex analysis textbook will give you the basic information about the subject we use here. If you would like to get some nice intuition about complex analysis, I suggest checking out [18].

Definition 10. Given a set X, a **topology** on X is a collection of subsets we call the *open sets* such that (1) the empty set and X are both open (2) finite intersections of open sets are open and (3) all unions of open sets are open.

Definition 11. A set X together with a topology is a **topological space**.

Definition 12. A **continuous** function between topological spaces is a function with the property that the inverse image of every open set is open.

Definition 13. A **homeomorphism** is a bijective continuous function between topological spaces with a continuous inverse.

Definition 14. A function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is **holomorphic** on the open set U if the function f is complex differentiable on every point p in U. Differentiability of a complex function is the same definition as for real functions, just with complex values instead of real values. If $U = \mathbb{C}$, then the function is sometimes called an **entire** function. The definition of holomorphic functions is very powerful as, unlike with real functions, being differentiable in the complex plane automatically guarantees the function is infinitely differentiable, and that the function is equal to its Taylor expansion (this last idea is called **analytic**).

Definition 15. A **Hausdorff** space is a topological space X in which we can separate all pairs of points by open neighborhoods, i.e. given any two distinct elements $x, y \in X$, there exist neighborhoods U of x and V of y so that $U \cap V = \emptyset$.

Definition 16. A topological space is **second countable** if there is a countable base for the topology. A **base** for a topology is a collection of open sets so that each open set in the topology can be written as a union of the sets in the base.

Definition 17. A topological space is **connected** if it cannot be written as the union of two disjoint nonempty open sets.

Definition 18. Given a topological space X, a **partition** P of X is a collection of subsets of X so that each element of X is in one, and only one, of the subsets. With the map $\pi : X \to P$ sending x to the subset of the partition P in which x lives, the set P inherits the topology of X by defining the open sets in P to be subsets U of P so that $\pi^{-1}(P)$ is open in X. In this paper, our P is X/G. We call this inherited topology the **quotient topology**.

8 Further Reading

My first introduction to Riemann surfaces was the book [16]. For my own research, I regularly refer to [6]. It has many important results in it but I would not recommend it as a first introduction to the field. In preparing this paper, I also read through [25]. Group actions are usually not covered extensively in undergraduate level abstract algebra books. If you are ready to read more about them, I recommend Section 1.7 and Chapter 4 in [4]. A reasonable introduction to algebraic geometry for undergraduates is [22]. Do not discount `wikipedia.org` as a reference for mathematical topics, although the mathematical entries can be dense.

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