A DATABASE OF GROUP ACTIONS ON RIEMANN SURFACES

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Abstract. The automorphism group of a Riemann surface is an important object in a number of different mathematical fields. An algorithm of Thomas Breuer determines all such groups for a fixed genus given a complete classification of groups up to a sufficiently large order, but data generated from this algorithm did not include the generators of the corresponding monodromy group, another crucial piece of information for researchers. This paper describes modifications the author made to Breuer's code to add the generators, as well as other new code to compute additional information about a given Riemann surface. Data from this project has been incorporated into the *L-functions and Modular Forms Database* (`http://www.lmfdb.org`) and we also describe the relevant data which may be found there.

1. Introduction

Groups acting on Riemann surfaces are important to a range of mathematical topics from the Galois theory of extensions of $\mathbb{C}(z)$ [Völklein, 1996], to Jacobian variety decompositions [Lange and Recillas, 2004],[Paulhus, 2008], to Galois covers of the projective line corresponding to Shimura varieties [Frediani et al., 2015], to questions about indecomposable rational functions [Fried, 1973]. Most of these topics utilize the generators of the monodromy group of the covering corresponding to the mapping $X \to X/G$ from a Riemann surface X to the orbit space of X by the group G acting on it.

Breuer created an algorithm and wrote computer code to determine all groups acting on Riemann surfaces of a given genus [Breuer, 2000]. He ran the code up to genus 48, and recorded the groups along with limited information about the ramification of the mapping $X \to X/G$. Within his code, generators of the monodromy group were also computed, but not recorded. We added functionality to Breuer's code to fully compute these generators, and wrote new code to compute additional information about Riemann surfaces. As this data will aid other researchers, we are creating a publicly visible, easily accessible database containing this data.

Enter the *L-functions and Modular Forms Database* (LMFDB), a huge database of mathematical objects. As an established database with a strong infrastructure, LMFDB is an ideal location to post this data. Part of its goal is to provide opportunities for unexpected connections between mathematical concepts. This paper describes the modifications we made to Breuer's code, as well as additional computations we use to generate data on LMFDB (such as which actions correspond to

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full automorphism groups, and which correspond to hyperelliptic curves). The relevant code may be found at `http://github.com/jenpaulhus/group-actions-RS` and the database is at `http://www.lmfdb.org/HigherGenus/C/Aut`.

Section 2 is an overview of the necessary mathematical background on groups acting on Riemann surfaces, and in Section 3 we describe the theoretical underpinnings of the original code of Breuer. In Section 4 we explain the new mathematical information added to the data and discuss the organization of the data on LMFDB. Finally, in Section 5 we enumerate planned future additions to the database.

2. Background on Riemann Surfaces

Let X be a compact Riemann surface of genus $g \geq 2$ (also referred to as a "curve"), and let $G = \mathrm{Aut}(X)$, the group of biholomorphic maps from X to itself. It is well known that this group is finite and bounded in size by $84(g-1)$. There is a natural mapping $\phi : X \to Y = X/G$ where Y is the orbit space of X under the action of G (ϕ sends $x \in X$ to the orbit of x under the action of G), and g_0 denotes the genus of the quotient Y. It is possible that this mapping branches at several points of Y, say on a set $\mathcal{B} \subset Y$ of size r. Letting $\phi^{-1}(\mathcal{B}) \subset X$ be the inverse image of these points, the mapping from $X - \phi^{-1}(\mathcal{B})$ to $Y - \mathcal{B}$ is a degree d covering for some positive integer d. For details on the covering space theory used in the paper, we recommend [Lee, 2011, Chapters 11 and 12]. For our specific situation, we recommend [Fried, 1980] or [Breuer, 2000].

Fix a base point $y_0 \in Y - \mathcal{B}$. Then $\phi^{-1}(y_0)$ consists of d points in $X - \phi^{-1}(\mathcal{B})$, say $\phi^{-1}(y_0) = \{x_1, \ldots, x_d\} \subset X$. Now consider a loop starting at y_0 and traveling once around one branch point in $\mathcal{B}$. For each element x_i in $\phi^{-1}(y_0)$ this loop lifts uniquely to a path in X which starts at x_i and ends at some $x_j \in \phi^{-1}(y_0)$, thus defining a permutation on the d elements of $\phi^{-1}(y_0)$: send i to the number of the endpoint of the corresponding lift starting at x_i. There is one such permutation for each element of $\mathcal{B}$ and these r permutations induce a map $\rho : \pi_1(Y - \mathcal{B}, y_0) \to S_d$ where S_d is the symmetric group on d elements, and the image of ρ is called the *geometric monodromy group* which is isomorphic to $\mathrm{Aut}(X)$ in the case of Galois covers. The order of each permutation corresponding to a loop around one element of $\mathcal{B}$ is denoted m_i for $1 \leq i \leq r$. When X and Y are connected, the image of ρ is a transitive subgroup of S_d.

The universal cover of a compact Riemann surface is the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ which has automorphism group $\mathrm{PSL}(2, \mathbb{R})$, and so X may be described as the orbit space of $\mathbb{H}$ by a torsion free subgroup of $\mathrm{Aut}(\mathbb{H})$ (see [Breuer, 2000, Theorem 3.9] or [Jones and Singerman, 1987, 4.19.8]). Call that torsion free subgroup K. It is isomorphic to $\pi_1(X, x_0)$.

Similarly, Y is equivalent to the orbit space of $\mathbb{H}$ by a subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ called a *Fuchsian group*. These Fuchsian groups have an explicit presentation [Breuer, 2000, Theorem 3.2]:

$$\Gamma = \langle \alpha_1, \beta_1, \ldots, \alpha_{g_0}, \beta_{g_0}, \gamma_1, \ldots, \gamma_r \mid \prod_{i=1}^{g_0} [\alpha_i, \beta_i] \prod_{j=1}^{r} \gamma_j = 1, \gamma_j^{m_j} = 1 \rangle \tag{1}$$

where $[\alpha_i, \beta_i]$ is the commutator of α_i and β_i. The list of non-negative integers $[g_0; m_1, \ldots, m_r]$ is called the *signature* of Γ and is uniquely determined for each Fuchsian group. The action of Γ on $\mathbb{H}$ induces an action of Γ/K on $\mathbb{H}/K$, so

$G \cong \Gamma/K$. As such we have an exact sequence

$$1 \to K \xrightarrow{\iota} \Gamma \xrightarrow{\eta} G \to 1. \tag{2}$$

Then, $G = \text{Aut}(X)$ may also be defined as the image of a *surface kernel epimorphism*, a surjection $\eta : \Gamma \to G$. Observe that different surface kernel epimorphisms may exist for fixed groups Γ and G so to classify actions it is not sufficient to only give the group and signature. We also need to describe the map η via, say, a description of where η sends the generators. Due to the structure of Γ, the group G is completely defined by $2g_0$ *hyperbolic* generators $a_1, b_1, \ldots, a_{g_0}, b_{g_0}$ and r *elliptic* generators $c_1, \ldots, c_r$ such that the c_i have order m_i and the product $\prod_{i=1}^{g_0}[a_i, b_i] \prod_{j=1}^{r} c_j = 1_G$ where 1_G is the identity element of G. We call this list of $2g_0 + r$ generators of G a *generating vector*.

Conversely, suppose G is any transitive subgroup of some symmetric group S_d with $2g_0 + r$ generators $\{a_1, b_1, \ldots, a_{g_0}, b_{g_0}, c_1, \ldots, c_r\}$ such that the c_i have order m_i and $\prod_{i=1}^{g_0}[a_i, b_i] \prod_{j=1}^{r} c_j = 1_G$. We say such a group has *product one generators*, and a set of $2g_0 + r$ generators is a *product one generator*. Then any surjection $\eta : \Gamma \to G$ defined as $\eta(\alpha_i) = a_i$, $\eta(\beta_i) = b_i$, and $\eta(\gamma_i) = c_i$ has a corresponding kernel K, and G acts on the compact Riemann surface X defined as the orbits of K acting on $\mathbb{H}$.

Hence there is a one-to-one correspondence between surjective maps $\eta : \Gamma \to G$ with $\ker(\eta)$ a torsion free group and finite groups which have product one generators. This is the beautiful existence theorem of Riemann (really a generalization of it) and it gives a way to translate the topological language of ramified coverings to the world of generators of finite groups. There are several very good sources on Riemann's existence theorem, particularly [Fried, 1980]. For a brief survey with generalizations and historical perspectives, see [Harbater, 2015]. The topic is also treated briefly in [Miranda, 1995, pg. 90-94], or in relation to function fields and the Inverse Galois Problem in [Völklein, 1996].

As with most mathematical objects, many unequal surface kernel epimorphisms exhibit identical behaviors. For example, relabeling the elements of $\phi^{-1}(y_0)$ (or reordering the c_i) should not constitute creating a "new" action. There are a number of different equivalence relations that may be placed on the surface kernel epimorphisms and we must make choices about which equivalence relation to classifying group actions up to in the database. For more information on classifications of automorphism groups of Riemann surfaces up to other equivalence classes see Sections 4 and 5. Breuer's algorithm computes epimorphisms up to an equivalence relation which is slightly weaker than topological or conformal equivalence, meaning two distinct group actions in his data may actually be topologically (or even conformally) equivalent.

Let G be a finite group which is the image of a surface kernel epimorphism $\eta : \Gamma \to G$, with $[g_0; m_1, \ldots, m_r]$ the signature of Γ. We denote by $\mathcal{C} = (C_1, \ldots, C_r)$ a list of r conjugacy classes in G (not necessarily distinct) each containing elements of order m_i. Define S to be the set $\{(s_1, \ldots, s_r) : s_i \in C_i\}$. Then G acts on S by component-wise conjugation called *simultaneous conjugation*. We note for later that this is precisely the action of the inner automorphisms of G on the generating vectors.

The properties of a tuple in S being a product one generator are invariant under simultaneous conjugation. In the special case when these tuples are generating vectors, any two vectors in the same orbit under simultaneous conjugation represent conformally equivalent actions in the Riemann surface (although the converse is not always true). This follows from the definition of conformal equivalence (see Section 5.2) and the fact that conjugation is an element of $\mathrm{Aut}(G)$. We classify our actions up to simultaneous conjugation.

Given a Riemann surface X of genus g, a group G acting on X, a tuple $\mathcal{C} = (C_1, \ldots, C_r)$ of conjugacy classes of G, and a generating vector $(s_1, \ldots, s_r)$ with s_i in C_i, then the tuple $(g, G, \mathcal{C})$ is called a *refined passport* [Sijsling and Voight, 2014] (alternatively that X *is of ramification type* $(g, G, \mathcal{C})$ [Magaard et al., 2002]). A *passport* is a similar tuple of information, but the conjugacy classes are only considered in S_d, so the actions are only classified up to the cycle type of the generators of G.

3. Breuer's Code

Breuer's contribution to this topic was to devise an algorithm to generate a list of all groups and corresponding signatures for which there is a surface kernel epimorphism $\eta : \Gamma \to G$ for a fixed genus. We only give a brief overview of his algorithm here (see [Breuer, 2000] for more details).

Breuer's algorithm first generates a list of all possible signatures for Fuchsian groups Γ for a given genus g and given order n of the automorphism group, using combinatorial restrictions on possible m_i values, as well as the Riemann-Hurwitz formula.

Next the algorithm searches the small group database in [GAP, 2006] and uses group theoretic results to construct a list of groups G of order n which could have one of the determined admissible signatures for that n. If a group of order n does not have elements of orders corresponding to the values in the signature, it is removed from the list of potential automorphism groups.

Finally, the algorithm determines which possible groups G satisfy the condition that there is a surjective morphism $\eta : \Gamma \to G$. This step in the algorithm utilizes several different group theoretic results concerning the structure of conjugacy classes. The algorithm first attempts to show no such surjection exists. It determines all possible lists of conjugacy classes $\mathcal{C} = (C_1, \ldots, C_r)$ such that the order of elements in C_i is m_i (i.e., potential refined passports for a given genus and group). Breuer then computes the size of $\mathrm{Hom}_{\mathcal{C}}(g_0, G)$, the set of homomorphisms from the Fuchsian group corresponding to the given signature to the group G, using the following theorem.

Theorem 3.1 (Theorem 3, [Jones, 1995])**.** *With* $\mathcal{C} = (C_1, \ldots, C_r)$ *as above,*

$$|Hom_{\mathcal{C}}(g_0, G)| = |G|^{2g_0-1} \sum_{\chi \in Irr(G)} \chi(1)^{2-2g-r} \prod_{i=1}^{r} \sum_{\sigma_i \in C_i} \chi(\sigma_i).$$

When this value is 0, there cannot be a surface kernel epimorphism for that refined passport. In the case where $g_0 = 0$ a result in [Scott, 1977, Theorem 1] gives a sufficient condition on the irreducible characters of a group G to show there is not a surjective homomorphism $\eta : \Gamma \to G$.

Conversely, to show there is an epimorphism $\eta : \Gamma \to G$, a specific generating vector defining the particular surface kernel epimorphism must be found (as the images in G of $\alpha_i, \beta_i, \gamma_j$ from (1) under the mapping η). A brute search of all possible generating vectors for a given refined passport is not feasible, especially for large signatures or large groups.

Instead Breuer uses the following proposition to quickly generate one element of each orbit under the action of simultaneous conjugation.

Proposition 3.2 (Lemma 15.27, [Breuer, 2000])**.** *Fix elements $\sigma_i \in C_i$ for each $1 \leq i \leq r$. Then the following set T gives us precisely one representative for each orbit of the action of G on $S = \{(s_1, \ldots, s_r) : s_i \in C_i\}$ by simultaneous conjugation:*

$$T = \{(\sigma_1, \sigma_2^{b_2}, \ldots, \sigma_r^{b_r}) : b_i \in R(b_1, \ldots, b_{i-1}) \text{ for } 2 \leq i \leq r\}$$

where $R(b_1, \ldots, b_{i-1})$ is a set of representatives of the double coset

$$C_G(\sigma_i)\backslash G/C_G(\sigma_1, \sigma_2^{b_2}, \ldots, \sigma_{i-1}^{b_{i-1}}),$$

defined iteratively and where $C_G(g_1, g_2, \ldots, g_k)$ means the intersection of the centralizers of $g_i \in G$ for $1 \leq i \leq k$.

Each element of T is tested to see if it is a product one generator. Breuer did not record these generating vectors in his original data, though. His goal was to list group and signature pairs only.

4. New additions

As mentioned above, one way to fully classify group actions on Riemann surfaces, is to produce a generating vector for each action. We converted Breuer's code to the computer algebra language Magma [Bosma et al., 1997] to align the code with other programs written by the author. We also added functionality which, given a group and signature, outputs the generating vector(s) for each refined passport up to simultaneous conjugation, generated via Proposition 3.2 (see [Paulhus, 2015], specifically the file `genvectors.mag`). With this code we do not need to reproduce all of Breuer's program. We use his already generated group and signature pairs as a starting point, and then add the generating vectors using the modified version of his code.

There is a software package in GAP called `MapClass`, which, among other computations, finds the generating vectors given a group and list of conjugacy classes corresponding to a refined passport [James et al., 2012]. Quotients of all triangle groups (actions such that $g_0 = 0$ and $r = 3$ or 4) acting on surfaces of genus up to 101, giving one generating vector per group and signature pair may be found at [Conder, 2007]. We also note that lists of actions with monodromy up to genus 21 were independently computed and posted online [Karbas and Nedela, 2013].

4.1. **Full Actions.** One important piece of information which is not determined in Breuer's original code is whether the group action described is the full automorphism group for the family of curves with corresponding data. Suppose we have an exact sequence

$$1 \to K \xrightarrow{\iota} \Gamma \xrightarrow{\eta} G \to 1$$

as in (2), and a corresponding generating vector from our modified version of Breuer's code. It is possible that there is some group H, Fuchsian group Γ_0 so that $G < H$, a mapping $j : \Gamma \to \Gamma_0$, and an exact sequence

$$1 \to K \xrightarrow{\iota_o} \Gamma_0 \xrightarrow{\eta_0} H \to 1$$

so that $\eta = \eta_0 \circ j$. In this case, the generic element of this family of Riemann surfaces has automorphism group H and signature that of Γ_0.

In [Ries, 1993] there are conditions for determining exactly when this situation occurs. (Identical results were independently discovered in [Bujalance et al., 2003].) Given G and Γ, the paper also describes explicitly how to compute H and Γ_0. The cases where $G \lhd H$ are covered in [Ries, 1993, Theorem pg. 390], while the remaining cases are covered in Table 1 and Table 2 of that paper. First, the signature of Γ must match one of only a handful of signatures for which this scenario can happen. For example, if $g_0 = 0$ and there are more than 4 branch points, the given group G is always the full automorphism group of the generic point of the family (η in this case never satisfies the conditions outlined in [Ries, 1993]). In the cases where $G \lhd H$, if the signature is one of the few that might lead to a larger automorphism group there must also exist an element of the automorphism group of G that behaves in a certain way on the generating vector corresponding to the action η.

We have written code [Paulhus, 2016] which takes the output of the modified Breuer program and determines if the mapping η defined by a generating vector satisfies one of the conditions outlined in Ries. When such an example is found, the group H and signature of Γ_0 are also recorded. One caveat: the code only determines the group H and signature of Γ_0, it does not determine exactly which refined passport (if there is more than one) the original group G and signature correspond to. This should be possible to determine using information in the proof of Theorem pg. 390 in [Ries, 1993].

In the special case when the signature of the action is $[0; k, k, k]$ or $[0; k, k, k, k]$, we must determine if there exists an automorphism of G which acts in a certain way on a generating vector *up to applying an element of* $Aut^+(\Gamma)$ *to the elements of the generating vector*, where $\mathrm{Aut}^+(\Gamma)$ is orientation preserving automorphisms of Γ. In the two cases when this happens, $g_0 = 0$ so the group $\mathrm{Aut}^+(\Gamma)$ is the Artin braid group. This group is an infinite (but finitely generated) group generated by $Q_1, \dots, Q_{r-1}$ where Q_i is the mapping sending one generating vector $(s_1, s_2, \dots, s_r)$ to $(s_1, \dots, s_{i-1}, s_{i+1}, s_{i+1}^{-1} s_i s_{i+1}, s_{i+2}, \dots, s_r)$ [Magnus et al., 1966, Section 3.7]. We call two generating vectors which are equivalent up to the action of this group *braid equivalent*.

Even though the braid group is infinite, the orbit of a given generating vector under the action of the elements of the braid group is finite (since the group G is finite there are only a finite number of generating vectors). To exhaustively determine whether the action corresponds to the full group, we need to generate the whole orbit of a given generating vector and test if there is an element of $\mathrm{Aut}(G)$ which acts on one of the generating vectors in that orbit in such a way to satisfy the conditions as described in Ries's paper. To do this, given a generating vector and all cycles of it (or permutations if the group is abelian), we apply the braids Q_1, Q_2 (and Q_3 in the case of $[0; k, k, k, k]$) to the list of generating vectors and test all of the elements in this list against the condition set out in [Ries, 1993, Theorem pg. 390]. If we find an automorphism satisfying the conditions in this theorem, we have a candidate for the full automorphism group. If not, we apply the braids to the new larger set and repeat the process. Eventually the whole orbit is generated (if it doesn't find, along the way, a generating vector in the orbit which satisfies the condition mentioned above) and the program will terminate since the orbit is finite. If it terminates without finding a generating vector satisfying the conditions, the action represented by the initial generating vector must be the full automorphism group.

4.2. **Special Properties.** Once we determine whether an action represents the full automorphism group, we compute additional information connected to the given refined passports. For example, Riemann surfaces described by these actions might be hyperelliptic curves or cyclic trigonal curves. A hyperelliptic curve of genus g is defined by the presence in its automorphism group of a central involution with $2g+2$ fixed points, while a cyclic trigonal curve of genus g is defined by the presence of an automorphism of order 3 which fixes $g+2$ points. Using [Swinarski, 2018], given a generating vector we compute the number of fixed points of a given automorphism (also see [Breuer, 2000, Lemma 10.4]), and then determine if the curve is hyperelliptic or cyclic trigonal. The code also computes the hyperelliptic involution or trigonal automorphism, which we include in the database.

Work of the author gives a method to use the automorphism group of a curve (and the generating vectors of the action) to produce a decomposition of its Jacobian variety [Paulhus, 2008]. The code to implement this method may be found at [Paulhus and Rojas, 2016] and we use that code on our compiled list of generating vectors. An entry such as $E \times E^3 \times A_4 \times A_5^2$ in the database means the decomposition consists of four factors: an elliptic curve, three isogenous copies of (possibly) another elliptic curve, one dimension four abelian variety, and two isogenous copies of a dimension five abelian variety. Each factor corresponds to a particular irreducible $\mathbb{C}$-representation of G and we also record the corresponding irreducible $\mathbb{C}$-character as determined by Magma's character table for the group.

While generating vectors themselves are enough to classify group actions on Riemann surfaces, the equation(s) for the curves in a given family are valuable to know as well. Determining an equation for a curve given an automorphism group and signature is, in general, a very hard problem. Equations are known for hyperelliptic curves [Shaska, 2003], genus 3 curves with automorphisms [Magaard et al., 2002], and genus 4-7 curves with "large" automorphism groups (the size of the automorphism group is at least $4(g-1)$) [Swinarski, 2018]. We added all these equations to the data with one small exception. In [Shaska, 2003] the equations are classified up to passports, not up to refined passports (the cycle structure of the generating vectors instead of the conjugacy classes in G). In two cases (if $G \cong C_2 \times C_2$, and if $G \cong C_4 \times C_2$ and the quotient of G by the hyperelliptic involution is $C_2 \times C_2$) there is more than one equation listed in [Shaska, 2003] but in our data there are distinct refined passports which are in the same passport. The author does not know a way to determine which equation(s) correspond to which refined passport.

4.3. **Equivalence Relations.** As we mentioned earlier, distinct generating vectors may well produce actions which are the same up to certain equivalence relations. Breuer's code already only produces actions up to simultaneous conjugation, but we also compute equivalence classes for two other equivalence relations.

Two actions η_1 and η_2 are *topologically equivalent* if there exists an $\omega \in \mathrm{Aut}(G)$ and $\phi \in \mathrm{Aut}^+(\Gamma)$ so that the following diagram commutes [Broughton, 1991].

$$\begin{array}{ccc} \Gamma & \xrightarrow{\eta_1} & G \\ \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \omega} \\ \Gamma & \xrightarrow{\eta_2} & G \end{array}$$

Notice this means that $\eta_2 = \omega \circ \eta_1 \circ \phi^{-1}$ where ϕ is an element of $\mathrm{Aut}^+(\Gamma)$ and $\omega \in \mathrm{Aut}(G)$. As such, two actions are topologically equivalent precisely when they are in the same orbit under the action of $\mathrm{Aut}(G) \times \mathrm{Aut}^+(\Gamma)$ [Broughton, 1991, Proposition 2.2]. This last statement translates the definition of topological equivalence to an algebraic condition which is computationally feasible to check in many cases. Based on Sage code described in [Behn et al., 2020] we wrote Magma code which, in the case when $g_0 = 0$, inputs all generating vectors (up to simultaneous conjugation) for a fixed group and signature and returns a representative (and the corresponding orbit) of each equivalence class of generating vectors. We restrict to $g_0 = 0$ because $\mathrm{Aut}^+(\Gamma)$ is much easier to work with in this case.

In the study of Hurwitz spaces (and the related inverse Galois problem) generating vectors up to the action of $\mathrm{Inn}(G) \times \mathrm{Aut}^+(\Gamma)$ are instead used. Since Breuer's code already computes one representative per equivalence class under the action of inner automorphisms (which is precisely simultaneous conjugation) and since the actions of each group in this direct product commute with each other, to find the orbits under the action we only need consider the action of $\mathrm{Aut}^+(\Gamma)$ on the output of the modified Breuer code for each group and signature pair. When $g_0 = 0$, this action is exactly the *braid action* we described in Section 4.1 and we use the same technique described there to compute equivalence classes under the braiding action and assign a representative generating vector for each orbit.

4.4. **Summary.** One note about our presentation of groups. Breuer's original code outputs a group as labeled in Magma or GAP, so as a pair (a, b) which indicates the group is of order a and is the bth group of that order in the database of small groups. Our Magma version of Breuer's code requires the group to be a permutation group to compute double coset representatives as in Proposition 3.2. However, in Magma many groups of the form `SmallGroup(a,b)` are not permutation groups. Also, to correspond to the mapping $\rho : \pi_1(Y - \mathcal{B}, y_0) \to S_d$ from Section 2, the group G must be transitive and satisfy the Riemann-Hurwitz formula. So we first convert the group to a permutation group, as in the standard proof of Cayley's theorem. The code to do this is at [Paulhus, 2016]. In doing so, we are specifying that our covers are Galois.

Putting everything together, the final process to create the database at `http://www.lmfdb.org/HigherGenus/C/Aut` is:

- For a fixed genus, load all the signature and group pairs computed with Breuer's original program and loop over this data.
- Convert groups of the form `SmallGroup(a,b)` in Breuer's data to permutation groups.
- Use our modified version of Breuer's code to determine the refined passports, and compute generating vector(s) for each.

- Determine if the action on each refined passport describes the full automorphism group of the family.
- Compute the Jacobian variety decomposition.
- If the action is the full action, check if the family consists of hyperelliptic or cyclic trigonal curves. In special cases we add equations.
- In the case of $g_0 = 0$, determine equivalence classes and representatives up to braid and topological equivalence.
- Future additional information will be computed at this point.

4.5. **Organization of the data on LMFDB.** As of publication of this paper, the database contains complete data up to genus 15 when the quotient X/G is the Riemann sphere ($g_0 = 0$) and up to genus 7 when $g_0 > 0$.

Each tuple of information: (genus, group, signature) has its own page on LMFDB. On each such page there is a list of the different refined passports corresponding to the given genus, group, and signature, and links to individual pages for each refined passport. Up to genus 7, every page also gives an option to only view actions up to topological equivalence. Clicking on the label of the given representative for an equivalence class leads to a page which lists all the refined passports in the given equivalence class (and further delineated according to which are braid equivalent to each other).

The individual pages of each refined passport list all generating vectors corresponding to this passport. We also list which conjugacy classes the refined passport corresponds to (as labeled by Magma when we initially generate the data–see Section 5.1). These pages also contain information about whether the action represents the full automorphism group of the family of Riemann surfaces. If the example is not the full automorphism group, a link to the action which does correspond to the full automorphism group is also included. We note if a refined passport of a full automorphism group corresponds to a hyperelliptic curve or a cyclic trigonal curve, and list the corresponding hyperelliptic involution or trigonal automorphism. Known equations are also displayed on these pages. Up to genus 7 if there is more than one generating vector on a page, there is an option to list only the representatives of each orbit under the braid action instead of all generating vectors. This feature is of particular value as the genus gets large as there are examples of refined passports with thousands of distinct generating vectors up to simultaneous conjugation but only a small handful up to braid action.

On both types of pages, a download button is available which downloads a Magma or GAP record with information for the given refined passport (or several records representing all the refined passports corresponding to a specific group and signature). For researchers working on questions requiring computations of generating vectors this feature should be the most useful as these files can simply be downloaded and then loaded into Magma or GAP for immediate access to the generating vectors. Also, a variety of search fields such as signature, or dimension of the family, or whether the family is hyperelliptic add to the functionality of the pages, and all search results may also be downloaded as Magma or GAP files.

A variety of statistics about the data currently in the database reside at `https://www.lmfdb.org/HigherGenus/C/Aut/stats`. The statistics list the maximum order of a group acting for each genus and all the unique groups which act for a fixed genus. The number of distinct refined passports and distinct generating vectors for each genus are also calculated.

5. Future Work

We plan to add additional information to the database. Here are a few examples.

5.1. Better Representation of Groups. One issue with the current data is that different representations of isomorphic groups can create different lists of generating vectors as the representatives of each orbit under the equivalence relations we have discussed. Also the labeling of the irreducible characters or conjugacy classes is dependent on Magma's labeling for that particular representation of the group (so the 2nd conjugacy class may not represent the same conjugacy class for distinct isomorphic groups).

Recently a database of small groups up to order 2000 (except order 1024) has been incorporated into LMFDB (see `https://www.lmfdb.org/Groups/Abstract/`). Among many other pieces of information for each group, particular elements of the group are fixed as generators (as are relations defining the group) and the conjugacy classes and irreducible characters of the group have a fixed labeling, all assigned in a deterministic way.

We can redo the computations from scratch (i.e., follow the steps outlined in Section 4.4) but now starting from the fixed representation of the group as defined in the small group database. Doing so ensures that labeling of generating vectors, conjugacy classes, and irreducible characters will be deterministic. No more debate over what is meant by the 2nd irreducible character or the 2nd conjugacy class of the group! The group pages also produce character tables and we will be able to link the irreducible characters listed on our pages directly to the corresponding row of the character table presented on the group's page.

5.2. Equivalence Relations. Some researchers only requires knowledge about distinct actions up to conformal (or analytic) equivalence. Two actions $\eta_1 : \Gamma \to G$ and $\eta_2 : \Gamma \to G$ are *conformally equivalent* if there is some $\omega \in \mathrm{Aut}(G)$ and $\widetilde{h} \in \mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}(2, \mathbb{R})$ so that the following diagram commutes

$$\begin{array}{ccccc} K & \longrightarrow & \Gamma & \xrightarrow{\eta_1} & G \\ \Big\downarrow \widetilde{h}^* & & \Big\downarrow \widetilde{h}^* & & \Big\downarrow \omega \\ K & \longrightarrow & \Gamma & \xrightarrow{\eta_2} & G \end{array}$$

where $\widetilde{h}^*$ is the map that takes some $\gamma \in K$ (or in Γ) and sends it to $\widetilde{h}\gamma\widetilde{h}^{-1}$ [Broughton, 1991]. This definition induces a conformal mapping $h : X \to X$ where $X = \mathbb{H}/K$. We hope to find a way to efficiently compute equivalence classes of generating vectors up to conformal equivalence, and then provide options on the LMFDB pages to only show generating vectors up to conformal equivalence.

5.3. Higher Genus Data. Breuer computed all group and signature pairs up to genus 48, and Conder computed group and signature pairs for large groups (those with $|G| > 4(g-1)$) up to much higher genus [Conder, 2014]. We plan to use the steps described in Section 4.4 to compute and then upload higher genus data to the database, although first some current code will need to be made more efficient to effectively compute data in higher genus.

As one particular example, the code to compute orbits of actions under topological equivalence is very slow for particular families of groups as the genus increase. There are several theoretical results and computational techniques that will speed up these computations. In addition, for $g_0 > 0$ the action of $\mathrm{Aut}^+(\Gamma)$ is more complicated than in the case where the quotient genus is the Riemann sphere and so we don't currently provide the option to list actions with $g_0 > 0$ up to topological equivalence. The code we use to compute topological equivalence would need to be rewritten to be able to do so.

5.4. **Other topics.**

- There is much current research on superelliptic curves, and we could incorporate known data about these families into LMFDB.
- A new section in the LMFDB provides a database of Belyĭ maps [Musty et al., 2019]. There are many connections that could be made between that database and the one described in this paper.
- It is possible to compute information about intermediate quotients X/H for $H < G$ which could also be displayed for each action.
- The group and signature pairs which show up for a fixed genus create a poset. We could display such a diagram to emphasize connections among families of curves in the moduli space $\mathcal{M}_g$.
- The Riemann matrix and corresponding period matrix are crucial objects for understanding certain computational properties of Riemann surfaces.
- It would be nice to determine the fields of definition of these curves.

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