Math 7770 Spring 2011 Midterm
Due: March 7, 2011

You are to work alone on this assignment. You may consult our textbook, your notes from the course, old homeworks and their solutions, or your pets. You may also ask me for clarification on any problem. You are not to consult other textbooks, the internet, other professors, friends, family, etc. Make sure in your solutions you **completely** explain your answers and justify anything we have not already proven in class or on a homework.

Please solve 6 of the following problems. If you turn in extra solutions they will be considered extra credit. Clearly mark which 6 you want to have count otherwise I will assume the first 6 problems are for credit.

1. Which numbers have an odd number of divisors?

2. Twice a four-digit number has a remainder 1 upon division by 7, three times the number has a remainder 2 upon division by 8, and four times the number has a remainder 3 upon division by 9. What can this number be?

3. Let p be an odd prime and r be a primitive root modulo p.

   (a) If q is a primitive root modulo p, prove that rq is not.

   (b) If $q \in \mathbb{Z}$, $rq \equiv 1 \bmod p$, prove that q is a primitive root modulo p.

4. (a) Prove that if $\phi(n) = n - 1$, then n is prime.

   (b) Prove that there are infinitely many positive integers n for which $\phi(n) = n/3$.

5. Determine the length of the longest sequence of consecutive square free integers.

6. If $\gcd(m, n) = 1$ show that $m^{\phi(n)} + n^{\phi(m)} \equiv 1 \bmod mn$.

7. (a) Show if n has a divisor p such that $p \equiv 3 \bmod 4$ then there is no solution to $x^2 \equiv -1 \bmod n$.

   (b) Suppose that p and q are distinct primes satisfying p, $q \equiv 1 \bmod 4$. Show that the congruence $x^2 \equiv -1 \bmod pq$ has a solution.

8. For any positive integer n prove that

$$\sum_{d|n} \mu(d)\phi(d) = \prod_{p|n}(2 - p)$$

   where p is prime.