Math 7770 Spring 2011
Homework 4
Due: February 14, 2011

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books outside of our textbook or other people, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

Warmup

- §2.3 # 29. Characterize the set of positive integers n satisfying $\phi(2n) = \phi(n)$.
- §2.5 # 1. Suppose that $b \equiv a^{67} \bmod 91$ and that $(a, 91) = 1$. Find a positive number $\overline{k}$ such that $b^{\overline{k}} \equiv a \bmod 91$. If $b = 53$, what is $a \bmod 91$?
- Prove or exhibit a counter example for each of the following:
  (a) If $(m, n) = 1$ then $(\phi(m), \phi(n)) = 1$.
  (b) If n is composite, then $(n, \phi(n)) > 1$.
  (c) If the same primes divide m and n, then $n\phi(m) = m\phi(n)$.
- Prove that $\phi(n) > n/6$ for all n with at most 8 distinct primes.

Problems

1. §1.4 # 1. Use the binomial theorem to show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Then give a combinatorial proof of this.
2. §2.3 # 34. Prove that there is no solution of the equation $\phi(x) = 14$ and that 14 is the least positive even integer with this property. Apart from 14, what is the next smallest positive even integer n such that $\phi(x) = n$ has no solution?
3. §2.3 # 35. If n has k distinct odd prime factors, prove that $2^k \mid \phi(n)$.
4. Prove that $\sum_{d^2|n} \mu(d) = \mu^2(n)$ and more generally,

$$\sum_{d^k|n} \mu(d) = \begin{cases} 0 & \text{if } m^k \mid n \text{ for some } m > 1 \\ 1 & \text{otherwise.} \end{cases}$$

   The last sum is extended over all positive divisors d of n whose kth power also divide n.

Challenge

I. §2.3 #42. Find all positive integers n such that $\phi(n) \mid n$.