Math 7770 Spring 2011
Homework 10 (updated)
Due: April 11, 2011

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books outside of our textbook or other people, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

Warmup

- §9.4 # 1. Prove that the units of the rational number field $\mathbb{Q}$ are ± 1 and that integers α and β are associates in this field if and only if $\alpha = \pm\beta$.
- §9.5 # 6. Prove that the following assertion is false in $\mathbb{Q}(i)$: If $N(\alpha)$ is a rational integer, then α is an algebraic integer.
- §9.6 # 1. Prove that the units in $\mathbb{Q}(\sqrt{2})$ are $\pm(1+\sqrt{2})^n$ where n ranges over all integers. (You may freely use the ideas from Theorem 7.26).
- Show that $\mathbb{Z}[\sqrt{-5}]$ contains no element whose norm is 2 or 3.

Problems

1. §9.5 #4. If α and $\beta \neq 0$ are integers in $\mathbb{Q}(\sqrt{m})$, and if $\alpha \mid \beta$, prove that $\overline{\alpha} \mid \overline{\beta}$ and $N(\alpha) \mid N(\beta)$.
2. §9.5 # 7. Prove that the following assertion is false in every quadratic field: If $N(\alpha)$ is a rational integer, then α is an algebraic integer. (Hint: See the back of the book.)
3. Let D be the ring of integers in $\mathbb{Q}(\sqrt{m})$. Show that, given a rational integer $M > 0$ there are at most finitely many integers $\alpha \in D$ with $\max(N(\alpha), N(\overline{\alpha})) \leq M$.
4. If we let $\alpha = 3 + 4\sqrt{-1}$ then the equation $N(\alpha) = 5^2$ is another way of expressing the fact that $(3, 4, 5)$ is a Pythagorean triple. Using the identity $N(\alpha^n) = (N(\alpha))^n$ for $n = 2$ and $n = 3$, find two other triples. Let $\beta = 5 + 12\sqrt{-1}$ so that $N(\beta) = \beta^2$. Use α and β to find a triple with hypotenuse 65. Using $\gamma = 12 + 5\sqrt{-1}$ instead, find another triple with hypotenuse 65.

Challenge

I. If $R \subset S$ are integral domains, $\alpha \in S$ is said to be **integral over** R if $\alpha^m + b_1\alpha^{m-1} + \cdots + b_m = 0$ for suitable m, and b_i all in R. S is called **integral over** R if every element of S is integral over R. Prove that if S is integral over R then S is a field if and only if R is a field.

II. If k is any field containing a ring D, the set of all elements in k which are integral over D (see previous problem for definition) is called the **integral closure** of D in k. Show that the integral closure is a ring and that it is integrally closed.