MATH 232 Discrete Math
Homework 6

Basic Information
This assignment is due in Gradescope by 10 PM on the dates below.

Make sure you understand MHC honor code and have carefully read and understood the additional information on the class syllabus and the grading rubric. I am happy to discuss any questions or concerns you have!

You are always welcome to ask me for small hints or suggestions on problems.

Problems

Reading Problem 6M (Due: Sunday, October 19)

Suppose we want to prove that for all $n \in \mathbb{Z}^+$ we have $1 + 3 + 5 + \cdots + (2n-1) = n^2$ using induction. What is the base case? What is the induction hypothesis? (You do not have to prove the result.)

Wednesday Problems HW6 (Due: Wednesday, October 22)

Be sure for the proof problems that you use the techniques and proof-writing guidelines we have talked about in class.

1. For this problem , let A and B both be *finite* sets. Explain your answers in an informal way (as if you were explaining to a friend who doesn't take any math classes). I am not looking for a formal proof, but write your answers in several complete sentences.
   (a) If $f : A \to B$ is *injective*, what can we say about $|A|$ versus $|B|$? Why?
   (b) If $f : A \to B$ is *surjective*, what can we say about $|A|$ versus $|B|$? Why?
   (c) If $f : A \to B$ is *bijective*, what can we say about $|A|$ versus $|B|$? Why?

2. Determine whether each of the following relations is reflexive, symmetric, and transitive (you should check each individual property, not all three at once). If a certain property fails, you should give a specific counterexample.
   (a) $S = \mathbb{Z}$ where aRb means $a - b \neq 1$.
   (b) $S = \mathbb{Z}$ where aRb means that both a and b are even.
   (c) $S = \mathbb{Z}$ where aRb means $a \mid b$.

3. Let r and s be real numbers. Define rRs whenever $r - s$ is an integer. Prove that this is an equivalence relation and describe what the equivalence classes look like.

4. A friend tries to convince you that the reflexive property is redundant in the definition of an equivalence relation because they claim that symmetry and transitivity imply it. Here is the argument they propose:

   If $a \sim b$, then $b \sim a$ by symmetry, so $a \sim a$ by transitivity. This gives the reflexive property.

   Now you know that their argument must be wrong because one of the examples in Problem 2 is symmetric and transitive but not reflexive. Pinpoint the error in your friend's argument. Be as explicit and descriptive as you can.

5. Find the inverse of the function $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{n-1}{2} \text{ if } n \text{ is odd} \\ \frac{-n}{2} \text{ if } n \text{ is even} \end{cases}$$

and prove that you found the right function by showing $f^{-1} \circ f = \mathrm{id}_A$ and $f \circ f^{-1} = \mathrm{id}_B$.

6. Prove that for all $x, y \in \mathbb{Z}$ if $a \equiv b \mod m$ and $c \equiv d \mod m$ then $ax + cy \equiv bx + dy \mod m$.

7. In ordinary arithmetic, if $a^2 = b^2$, then $a = \pm b$. Is the analogous statement true in modular arithmetic, i.e. if $a^2 \equiv b^2 \mod m$ does that mean $a \equiv \pm b \mod m$? Either prove this or give a specific counterexample.

Reading Problem 6F (Due: Thursday, October 23)

Use the Pigeonhole Principle to prove that if 6 integers are chosen at random, then at least two of them will have the same remainder when divided by 5. (Hint: think about the Division Algorithm!)