Math 704 Summer 2008
Homework 7
Due: August 1, 2008 5:00 PM

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books or other people heavily, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

For problems 1-4 assume H and K are groups, ϕ a homomorphism from K into $\mathrm{Aut}(H)$, and identify H and K as subgroups of $G = H \rtimes_\phi K$.

1. Section 5.5 #1. Prove that $C_K(H) = \ker \phi$. [Hint: $C_K(H) = C_G(H) \cap K$].
2. Section 5.5 #16. Show that there are exactly 4 distinct homomorphism from Z_2 into $\mathrm{Aut}(Z_8)$. Prove that two of the resulting semidirect products are isomorphic to $Z_8 \times Z_2$ and D_{16}.
3. Section 5.5 # 18. Show that if H is any group then there is a group G that contains H as a normal subgroup with the property that for every automorphism σ for H there is an element $g \in G$ such that conjugaction by G when restricted to H is the given automorphism σ.
4. Section 5.5 # 21. Let p be an odd prime and let P be a p-group. Prove that if every subgroup of P is normal then P is abelian. [Hint: You may find Exercise 20 of section 5.5 useful.]
5. Section 6.2 # 14. Prove there are no simple groups of order 144.
6. Section 6.3 # 2. Prove that if $|S| > 1$ then $F(S)$ is non-abelian.
7. Section 6.3 # 4. Prove that every nonidentity element of a free group is of infinite order.
8. Exhibit all degree 1 complex representations of a finite abelian group. Deduce that the number of such representations equals the order of the group. [Hint: First decompose the abelian group into a direct product of cyclic groups.]
9. Let X be a finite set on which G acts and let ρ be the corresponding permuation representation and let χ_X be the character of ρ. Let $g \in G$ and show that $\chi_X(g)$ is the number of elements of X fixed by g.