Math 704 Summer 2008
Homework 5
Due: July 21, 2008

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books or other people heavily, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

1. Section 4.5 # 5. Show that a Sylow p-subgroup of D_{2n} is cyclic and normal for every odd prime p.

2. Section 4.5 # 20. Prove that if $|G| = 1365$ then G is not simple.

3. Section 4.5 #30. How many elements of order 7 must there be in a simple group of order 168?

4. Let G be a finite group that has exactly 50 Sylow 7-subgroups. Let P be one of the Sylow 7-subgroups and $N = N_G(P)$ be the normalizer of P in G. Show that N is a maximal subgroup of G. (Maximal means that for any other $H \leq G$, if $N \subseteq H \subseteq G$ then $H = N$ or $H = G$.)

5. Let G be a finite group and p a fixed prime number that divides the order of G. Define $G(p) = \{g \in G \mid |g| = p^r \text{ for some } r \geq 0\}$. Prove that $G(p)$ is a subgroup of G if and only if G has a normal Sylow p-subgroup.

6. Section 4.5 #32. Let P be a Sylow p-subgroup of H and let H be a subgroup of K. If $P \trianglelefteq H$ and $H \trianglelefteq K$, prove that P is normal in K. Deduce that if $P \in \mathrm{Syl}_p(G)$ and $H = N_G(P)$ then H also equals $N_G(H)$.

7. Section 5.1 # 1. Show that the center of the direct product is the direct product of the centers. Deduce that the direct product of groups is abelian if and only if each of the factors is abelian.

8. Section 5.1 # 11. Let p be a prime and let $n \in Z^+$. Find a formula for the number of subgroups of order p in the elementary abelian group E_{p^n}.

9. Section 5.1 #17.(c). Let $I = \mathbb{Z}^+$ and let p_i be the ith integer prime. Show that if $G_i = \mathbb{Z}/p_i\mathbb{Z}$ for all $i \in \mathbb{Z}^+$ then every element of the direct sum of the G_i's has finite order but their direct product has elements of infinite order. Show that in this example the direct sum is the torsion subgroup of the direct product. (See Homework 2, #1).