Math 704 Summer 2008
Homework 4
Due: July 7, 2008

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books or other people heavily, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

1. 3.4 # 8. Let G be a finite group. Prove that the following are equivalent: (See book for hint).
   (a) G is solvable
   (b) G has a chain of subgroups: $1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_s = G$ such that H_{i+1}/H_i is cyclic $0 \leq i \leq s-1$
   (c) all composition factors of G are of prime order
   (d) G has a chain of subgroups $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_t = G$ such that each N_i is a normal subgroup of G and N_{i+1}/N_i is abelian, $0 \leq i \leq t-1$

2. 3.5 # 9. Prove that the unique subgroup of order 4 in A_4 is normal and isomorphic to V_4.

3. 4.1 # 1. Let G act on the set A. Prove that if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$ then $G_b = gG_ag^{-1}$. Deduce that if G acts transitively on A then then kernel of the action is $\bigcap_{g \in G} gG_ag^{-1}$.

4. 4.2 #2. List the elements of S_3 as $1, (1\ 2), (2\ 3), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)$ and label these with the integers 1, 2, 3, 4, 5, 6 respectively. Exhibit the image of each element of S_3 under the left regular representation of S_3 into S_6.

5. 4.2 # 8. Prove that if H has finite index n then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n!$.

6. Find all conjugacy classes and their sizes in the following groups:
   (a) $Z_2 \times S_3$ (b) A_4.

7. 4.3 # 13. Find all finite groups which have exactly two conjugacy classes.

8. 4.3 #30. If G is a group of odd order, prove for any nonidentity element $x \in G$ that x and x^{-1} are not conjugate in G.

9. Let G be a group. If $\text{Aut}(G)$ is a cyclic group, show that G is abelian.