Math 704 Summer 2008
Homework 3
Due: June 30, 2008

You are welcome to work together but everyone needs to write up **distinct** solutions. If you use any books or other people heavily, please make sure to give them credit. Make sure your solutions are complete. If your handwriting is atrocious, I am happy to give you a basic introduction to LaTeX.

1. Let G be a group (not necessarily finite) and $H \leq G$ and $K \leq G$. Assume $m = |G : H|$ and $n = |G : K|$.
   (a) Prove that $|G : H \cap K|$ is finite and does not exceed mn.
   (b) If m and n are relatively prime, show that $|G : H \cap K| = mn$.
2. 3.1 #5. Use 3.1 #4 to prove that the order of the element $gN \in G/N$ is n where n is the smallest positive integer such that $g^n \in N$ (and gN has infinite order if no such positive integer exists). Give an example to show that the order of $gN \in G/N$ may be strictly less than the order of $g \in G$.
3. Let $\phi : G \to H$ be a homomorphism between finite groups G and H with relatively prime orders. Prove that ϕ is the trivial homomorphism (i.e. $\phi(g) = 1_H$ for all $g \in G$).
4. 3.1 #12. Let G be the additive group of real numbers, let H be the multiplicative group of complex numbers of absolute value 1 (the unit circle S^1 in the complex plane) and let $\phi : G \to H$ be the homomorphism sending r to $e^{2\pi i r}$. Draw the points on a real line which lie in the kernel of ϕ. Describe similarly the elements in the fibers of ϕ above the points $-1, i, e^{4\pi i/3}$ of H.
5. 3.1 #36. Prove that if $G/Z(G)$ is cyclic then G is abelian. (Hint: If $G/Z(G)$ is cyclic with generator $xZ(G)$ show that every element of G can be written in the form $x^a z$ for some integer $a \in \mathbb{Z}$ and some element $z \in Z(G)$).
6. 3.2 #4. Show that if $|G| = pq$ for some primes p and q (not necessarily distinct) then either G is abelian or $Z(G) = 1$. (Hint: Use 3.1 #36).
7. (a) 3.2 #22. Use Lagrange's Theorem in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$ to prove Euler's Theorem: $a^{\phi(n)} \equiv 1 \mod n$ for every integer a relatively prime to n.

   (b) 3.2 #23. Determine the last two digits of $3^{3^{100}}$ by determining $3^{100} \mod \phi(100)$ and using (a).
8. 3.3 #3. Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either $K \leq H$ or $G = HK$ and $|K : K \cap H| = p$.
9. 3.3 #7. Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$. (Hint: Draw the lattice).