Math 321: Foundations of Abstract Algebra

HOMEWORK 1 : DUE JANUARY 31

1. **#1.3 and # 1.6.** In each case, determine whether or not the given $\star$ is a binary operation on the given set S. If $\star$ is a binary operation on S, determine whether $\star$ is commutative and whether it is associative.

   (c) $S = \mathbb{R} \quad a \star b = \frac{a}{a^2+b^2}$

   (g) $S = \{1, -2, 3, 2, -4\} \quad a \star b = \mid b \mid$

   (i) S =the set of all 2×2 matrices with real entries, and if

   $$a = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} \text{ and } b = \begin{pmatrix} r_5 & r_6 \\ r_7 & r_8 \end{pmatrix}$$

   then

   $$a \star b = \begin{pmatrix} r_1 + r_5 & r_2 + r_6 \\ r_3 + r_7 & r_4 + r_8 \end{pmatrix}.$$

2. **#2.1.** Which of the following are groups? Why?

   (c) $\mathbb{R} - \{0\}$ under the operation $a \star b = |ab|$

   (d) The set $\{-1, 1\}$ under multiplication

   (h) $\mathbb{R} - \{1\}$ under the operation $a \star b = a + b - ab$

3. **#2.10.** Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where a, $b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Show that G forms a group under matrix multiplication.

4. **#3.4.** Let g be an element of a group $(G, \star)$ such that for some one element $x \in G$, $x \star g = x$. Show that $g = e$.

5. **#3.6.** Prove the Cancellation Laws (Theorem 3.6).

6. **#3.9.** Let $(G, \star)$ be a group. Show that $(G, \star)$ is abelian if and only if

   $$(x \star y)^{-1} = x^{-1} \star y^{-1} \text{ for all } x, y \in G.$$

Extra

1. Show that any finite group with 4 elements must be abelian.