Dihedral Group Notes

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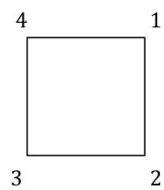
The **dihedral group** is a group formed from the plane symmetries of regular polygons. What are plane symmetries and regular polygons?

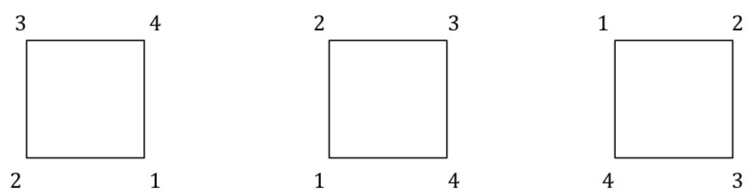
Definition. A **plane symmetry** of a figure is a function from the plane to itself that carries the figure onto itself and preserves distances.

Definition. A **regular polygon** is a polygon that is equiangular and equilateral (so all sides have the same length and all angles have the same measurement).

Example Let's start by considering the square. Label it with vertices 1, 2, 3, and 4 and assume the center of the square is the origin of a standard Cartesian plane.

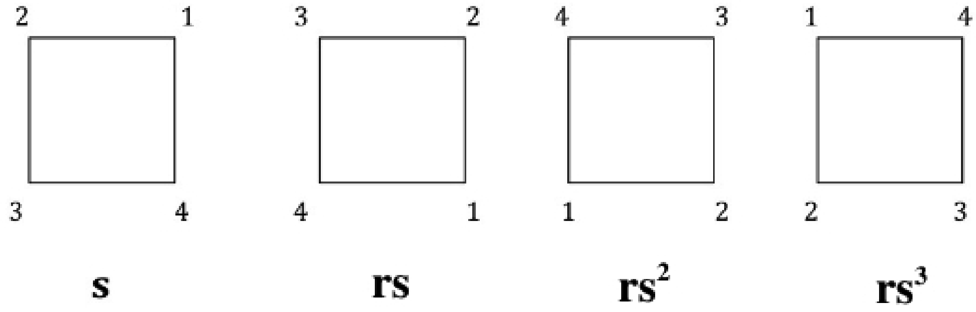


Notice that one plane symmetry is simply rotating the figure clockwise by 90° (or $\frac{\pi}{2}$ radians). We call that rotation r. We can also rotate by 180° (or π radians) and 270° (or $\frac{3\pi}{2}$ radians). Those three rotations are drawn below.



r **r²** **r³**

Another plane symmetry is reflection along the diagonal line connecting vertex 1 with vertex 3. We call this symmetry s. There is also reflection along the x-axis and y-axis, as well as the other diagonal. The four reflections are drawn below. Notice they are each compositions of s with one of the rotations above.



The pictures above represent all plane symmetries. Any other swapping of vertices you attempt will violate the distance preserving condition for plane symmetries. So, for instance, we cannot just switch vertices 1 and 4 above. This would change the distance between 1 and 3.

There are many relations among the pictures above. For instance, what happens if we reflect across the diagonal line from vertex 1 to vertex 3, followed by a 270° rotation? This is the same as doing one 90° rotation followed by reflection across the diagonal line from vertex 1 to vertex 3. In symbols $r^3s = sr$, we write this as with function composition. So r^3s means "apply s and then apply r^3".

Theorem 1. *The set of plane symmetries of a square under the operation of function composition forms a group called D_4 or* **the dihedral group on 4 objects** *(called the octic group in your book).*

Proof: The composition of plane symmetries must be a plane symmetry (it must preserve distance and carry the figure onto itself) and hence the operation is binary. Associativity follows from function composition. The identity is the trivial symmetry. And, finally, inverses exist. The intuitive idea is that one can "undo" any plane symmetry. For instance, s can be "undone" by another application of s, and r can be "undone" by applying r^3. □

In the discussion above, was there anything particularly special about squares? Could we do the same analysis with pentagon or hexagon or any regular n-gon? Definitely.

Definition. The **dihedral group of order** $2n$ is the group formed by the symmetries of a regular n-gon. We denote this group as D_n (although the occasional book or research paper will write this as D_{2n}).

Theorem 2. *Label the vertices of D_n starting with v_1 and working clockwise to v_2, v_3, etc. Let r be rotation of the n-gon by $2\pi/n$ radians and let s be reflection across the line connecting v_1 to the center of the object.*

(1) e, r, r^2, …,r^{n-1} are all distinct and $r^n = e$ so $o(r) = n$.
(2) $o(s) = 2$.
(3) $s \neq r^i$ for any i.
(4) $r^is \neq r^js$ for all $0 \leq i, j \leq n-1$ with $i \neq j$.

From this we can conclude that $D_n = \{e, r, r^2, \ldots, r^{n-1}, s, rs, r^2s, \ldots, r^{n-1}s\}$.

Proof: (1) Consider where v_1 gets mapped under each symmetry. The symmetry r sends v_1 to v_2, while r^2 sends v_1 to v_3 and r^i sends v_1 to v_{i+1} and $i + 1 \neq j + 1$ when $i \neq j$ and $0 \leq i, j < n$.

(2) Simply consider what applying s twice to each vertex will do to it. (3) The symmetry s fixes v_1 yet the only r^i which does this is r^n which is the identity e. However s is not the identity since it sends v_2 to v_n and so s is not a power of r. (4) Since $r_i \neq r_j$ by (1), reflecting each by s will not produce the same symmetry. □

Definition. Since every element of D_n is a product of s and r, we say that those two elements **generate** the group. In general we say that a subset S of a group G **generates** the group if every element of the group may be written as a product of elements in S.

Theorem 3. *Let $r,\ s \in D_n$ be as defined above.*
(1) $rs = sr^{-1}$.
(2) For homework you will prove that $r^i s = sr^{-i}$ for all $0 \leq i \leq n$.

Proof: For (1) consider where rs sends v_1. The symmetry s sends it to v_1, followed by the symmetry r which sends v_1 to v_2. Conversely, for sr^{-1} we first apply r^{-1} to v_1 which goes to v_n and then s sends v_n to v_2.

Similarly s sends v_2 to v_n and r sends v_n to v_1 while r^{-1} sends v_2 to v_1 and s preserves v_1.

In general, if $2 < i \leq n$ then s sends v_i to v_{n-i+2} and r sends v_{n-i+2} to v_{n-i+3} whereas r^{-1} sends v_i to v_{i-1} and s sends v_{i-1} to $v_{n-(i-1)+2} = v_{n-i+}$. So rs and sr^{-1} send every vertex to the same vertex which means they must be the same symmetry. □

Notice that (1) tells us that D_n is not abelian if $n \geq 3$. Theorem 3 above is useful for computations. For example if we want to know what $s(rs)$ is in the group, we can rewrite rs as sr^{-1} and get $s(rs) = s(sr^{-1}) = (ss^{-1})r^{-1} = r^{n-1}$ since s has order 2 and $r \cdot r^{n-1} = r^n = e$.