Math 218: Elementary Number Theory

Homework LAST!! : Due December 9

4.1 #7. Prove that the set of primes of the form $5+6k$ is infinite.

4.1 #8. Prove that the set of primes of the form $7+8k$ is infinite. (Hint: Choose $N = 2(n!)^2 - 1$ and you may assume Theorem 3.7.2 without proof.)

4.3 #6. Find an example of a pair of functions f, g so that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1 \text{ but } \lim_{n\to\infty} f(n) - g(n) = \infty.$$

Explain why your functions satisfy both statements. (Note: Perhaps start by thinking of calculus functions $f, g : \mathbb{R} \to \mathbb{R}$.) If you work with other students on the HW, you should each come up with distinct examples.

4.3 #7. (a) Prove that if p is a prime then $\frac{\pi(p-1)}{p-1} < \frac{\pi(p)}{p}$.

(b) Prove that if n is composite then $\frac{\pi(n-1)}{n-1} > \frac{\pi(n)}{n}$.

4.3 #8. Define the function $F : \mathbb{Z}^+ \to \mathbb{Z}$ as

$$F(n) = \left[\cos^2\left(\pi\frac{(n-1)!+1}{n}\right)\right]$$

where the outer brackets represent the *greatest integer function*.

(a) Prove that $F(n) = 1$ if n is prime or if $n = 1$, and $F(n) = 0$ if n is composite.

(b) Use (a) to prove that $\pi(n) = -1 + \sum_{i=1}^{n} F(i)$.