Math 215: Linear Algebra

WRITING ASSIGNMENT 3 : DUE DECEMBER 1

Make sure you are familiar with the Academic Honesty policies for this class, as detailed on the syllabus. All work is due on PWeb on the given day by 3 PM or by 7 PM if you $\LaTeX$ it.

This is a writing assignment. You should treat this like you would a writing assignment in an English or Philosophy or History course, in the sense that everything you write should be part of a complete sentence and part of a larger paragraph which serves a clear purpose, your grammar and spelling should be accurate, and (if you handwrite it) you should not have crossed out sections where you change your mind about what you want say. For each problem you should plan out how you want to write it in an outline or draft, and then write a polished, final product to be submitted. Your audience should be fellow students who are excited about math but have not learned linear algebra yet. You are welcome to ask a friend who is not in the class to read your answer and let you know if it makes sense to them. There is no page or paragraph limits for this assignment but you should be (1) thorough and complete and (2) concise and exact.

1. On PS 11, problem 4, you proved that for projection transformations $P_{\vec{w}}$ the only eigenvalues are 0 and 1. On PS 9, problem 2, you showed that, for one particular projection, $A \cdot A = A$. There is a relationship between these two facts.

   Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation which is not the identity transformation and is not the zero transformation. Carefully prove that T has eigenvalues 0 and 1 if and only if $T \circ T = T$.

2. Linear transformations which preserve the lengths of vectors are called *isometries*. In this problem you will prove that any linear transformation which is also an isometry has a standard matrix with several special properties.

   Assume $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and an isometry. An isometry means that $||T(\vec{v})|| = ||\vec{v}||$ for all $\vec{v} \in \mathbb{R}^2$. Also assume $[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and call $\vec{v_1} = \binom{a}{c}$ and $\vec{v_2} = \binom{b}{d}$.

   (a) Prove that the two columns of $[T]$ have magnitude 1, i.e. that $\left|\left|\binom{a}{c}\right|\right|$ and $\left|\left|\binom{b}{d}\right|\right|$ equal 1.

   (b) What is $\left|\left|\binom{1}{1}\right|\right|$?

   (c) Compute $T\left(\binom{1}{1}\right)$ using the standard matrix of T.

   (d) Use your answers from (a), (b) and (c) as well as the fact that T is an isoemtry to prove that $\langle \vec{v_1}, \vec{v_2} \rangle = 0$.