Solving Systems of Linear Equations

Math 215 Worksheet

1 Instructions

Remember what good partnerships look like: everyone in the group has good ideas, and you should make sure everyone understands a point before moving on. You will not turn these worksheets in (they are for your own benefit). **Fill in all of the bold questions below**.

2 Setup

First, **carefully read Definition 4.2.1 and 4.2.2.**

Write down your own example of a linear equation in at least 3 variables.

Write down a system of linear equations in at least 3 variables and with at least 3 equations.

Suppose we ask the following question:

$$\text{Is the Span}\left(\begin{pmatrix}1\\2\\-1\end{pmatrix},\begin{pmatrix}-3\\1\\2\end{pmatrix},\begin{pmatrix}-2\\-1\\1\end{pmatrix}\right)=\mathbb{R}^3? \tag{2.1}$$

This is the same as asking if the following *for all* statement is true (**fill in the blanks**):

For all $\vec{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\in$ ____________________, there are $x, y, z \in$ ____________________

so that ____.

Turn the last line on the previous page into a system of linear equations.

Question (2.1) from the previous page is equivalent to asking if the system of linear equations you wrote above has a solution. (Reread Definition 4.2.2 to make sure you know what a "solution" means.) Our goal today is to create a general method to find solutions to systems of linear equations to help us answer questions like (2.1).

3 Recap of $\mathbb{R}^2$

Consider three different systems from our previous work with $\mathbb{R}^2$:

$$\begin{aligned} 3x - y &= 4 \\ 2y &= 5 \end{aligned} \qquad \begin{aligned} 3x + 2y &= 4 \\ 3x + 2y &= 6 \end{aligned} \qquad \begin{aligned} 3x + 2y &= 4 \\ 6x + 4y &= 8 \end{aligned}$$

How many solutions do each of these systems have?

Similar situations can occur for $\mathbb{R}^n$. **How many solutions do each of these systems have?**

$$\begin{aligned} x + 2y - z &= 3 \\ y + z &= 4 \\ z &= 3 \end{aligned} \qquad \begin{aligned} x + 2y - z &= 3 \\ y + z &= 4 \\ y + z &= 1 \end{aligned} \qquad \begin{aligned} x + 2y - z &= 3 \\ y + z &= 4 \\ -2y - 2z &= -8 \end{aligned}$$

Now compute the solutions for each example (those with solutions). If there is more than one solution, your final answer should include as few variables as possible.

4 Elementary Row Operations

Notice how nice these "stepped" equations are to "read off" the solutions. Our goal will be to convert a system of equation like the one you wrote at the top of page 2 into one that looks "stepped". We use this method when we solve systems of two equations in two variables. When we multiply one equation by a real number to eliminate one variable, we are really just turning the system of linear equations into a "stepped" shape (and if you don't remember the system of eliminating one variable, check out page 51, paragraph 3 of the book).

To get to this "stepped" format, we apply operations called *elementary row operations.* **Read Definition 4.2.3 carefully** to see the three operations we are allowed to use. The key point of these three operations is that each is reversible, so we do not introduce any *new* solutions by applying these operations to the system. Hence the "stepped" system of equations will have the same solutions as the original system. (This is what Proposition 4.2.4 and 4.2.5 tell us.)

Let's use the example you produced from question (2.1), specifically is $\vec{v} = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$ in the span of the three vectors in (2.1)? Plugging $v_1 = -3$, $v_2 = -1$, and $v_3 = 0$ in for your equation at the top of page 2 should give you:

$$\begin{aligned} x - 3y - 2z &= -3 \\ 2x + y - z &= -1 \\ -x + 2y + z &= 0. \end{aligned}$$

We get rid of the x in rows 2 and 3 first. We will replace row 2 by the sum of row 2 and $(-2)\times$ row 1. This is the third operation in Definition 4.2.3. (Warning: Read this operation VERY carefully. You can't multiply row 2 by a real number and add that product to another equation, and then replace row 2. You could only replace that other equation in this example.) Similarly we can replace row 3 by the sum of row 3 and row 1 This gives us:

$$\begin{aligned} x - 3y - 2z &= -3 & & \\ 7y + 3z &= 5 & \qquad (-2)R_1 + R_2 &\to R_2 \\ -y - z &= -3. & R_1 + R_3 &\to R_3 \end{aligned}$$

Warning: As you solve these problems on homework and exams, I will expect some clear notation for each step. If you apply the first elementary row operation, you should write it as $3R_3 \to R_3$ to mean you are replacing row 3 with $3\times$ row 3. Similarly for the second operation, $R_2 \leftrightarrow R_1$ means swap row 1 and row 2, and for the third elementary row operation, $3R_1 + R_2 \to R_2$ means $3\times$ row 1 plus row 2 replaces row 2.

Back to our example, we now just need to get rid of that $-y$ in the last row to get a "stepped" shape. But wouldn't it be easier to get rid of the $7y$ in the second row? So let's swap row 2 and row 3 to get

$$\begin{aligned} x - 3y - 2z &= -3 & & \\ -y - z &= -3 & \qquad R_2 &\leftrightarrow R_3 \\ 7y + 3z &= 5 & & \end{aligned}$$

and then we can replace new row 3 with 7× row 2 plus row 3 to get

$$\begin{aligned} x - 3y - 2z &= -3 \\ -y - z &= -3 \\ -4z &= -16. & \qquad 7R_2 + R_3 \to R_3 \end{aligned}$$

Now solve this system of equations.

Is $\vec{v} = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix} \in$ **Span** $\left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right)$**? Why or why not?**

5 Matrices

Do we really need to carry the x, y, and z around everywhere? NO! So let's use matrices for bookkeeping purposes. **Read Definition 4.2.7** which defines the corresponding matrix of a system of linear equations, called the *augmented matrix*. Here is the augmented matrix for the example we have been working on:

$$\begin{pmatrix} 1 & -3 & -2 & -3 \\ 2 & 1 & -1 & -1 \\ -1 & 2 & 1 & 0 \end{pmatrix}.$$

And here is how the elementary row reductions would look for this example with matrices:

$$\begin{pmatrix} 1 & -3 & -2 & -3 \\ 2 & 1 & -1 & -1 \\ -1 & 2 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & -2 & -3 \\ 0 & 7 & 3 & 5 \\ 0 & -1 & -1 & -3 \end{pmatrix} \quad \begin{matrix} (-2)R_1 + R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{matrix}$$

$$\to \begin{pmatrix} 1 & -3 & -2 & -3 \\ 0 & -1 & -1 & -3 \\ 0 & 7 & 3 & 5 \end{pmatrix} \; R_2 \leftrightarrow R_3 \; \to \begin{pmatrix} 1 & -3 & -2 & -3 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & -4 & -16 \end{pmatrix} \; 7R_2 + R_3 \to R_3$$

At this point we can return the matrix to equations to get (again):

$$
\begin{aligned}
x - 3y - 2z &= -3 \\
-y - z &= -3 \\
-4z &= -16.
\end{aligned}
$$

which you found the solutions to above.

For the moment, don't think of the matrices as representing linear transformations, but simply as a shorthand way to carry around all the information we need to solve systems of equations.

Read Definitions 4.2.8, 4.2.9, and 4.2.10 as we will use the terminology from those definitions.

6 Some Examples

Now it is your turn to practice. On the next two pages are two examples to try. **Find all solutions to these systems by converting them to an augmented matrix and using elementary row operations to get the "stepped" version**. Clearly mark what operations you are doing at each step like I did above.

Example:

$$
\begin{aligned}
x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\
-x_1 - x_2 + 4x_3 - x_4 &= 6 \\
-2x_1 - 4x_2 + 7x_3 - x_4 &= 1
\end{aligned}
$$

Could you do more elementary row operations to make the solutions even easier to read off?

Example:

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 &= 0 \\
2x_1 + 3x_2 - x_3 - x_4 &= 2 \\
3x_1 + 2x_2 + x_3 + x_4 &= 5 \\
3x_1 + 6x_2 - x_3 - x_4 &= 4
\end{aligned}
$$