Math 215: Linear Algebra

Problem Set 7 : Due November 13

(18 points) Make sure you are familiar with the Academic Honesty policies for the class, described on the syllabus. All work is due on the given day by 3 PM Grinnell Time, or 7 PM if you LaTeX it.

1. (4 points) In the proof of Theorem 2.3.10 from class, we proved that *if (1) then (2)* is true when $a \neq b_1$, but we did not prove the case where $a = c$. Write a very similar proof to the one in class, except now start the proof by saying "Let $a = c$. Fix $\vec{v} = \binom{a-1}{c+1} \ldots$." (You should comment about why we don't have to worry about the case where a and c are both 0.)

2. (3 points) For this problem, we will prove one part of Proposition 2.3.8 from the textbook, namely:

   Let $\vec{u_1}$ and $\vec{u_2} \in \mathbb{R}^2$. If $Span(\vec{u_1}, \vec{u_2}) = Span(\vec{u_1})$, then $\vec{u_2} \in Span(\vec{u_1})$.

   I have set up the outline of the proof. You should either fill in the blanks or you may write your own proof from scratch, but it should look similar to my outline. **Please underline or color differently the filled in blanks on your submitted answer.**

   Assume that $\text{Span}(\vec{u_1}, \vec{u_2}) = \text{Span}(\vec{u_1})$. Notice that $\vec{u_2} =$ ________________________. Since ________________________ $\in \mathbb{R}$, it follows that $\vec{u_2} \in \text{Span}(\vec{u_1}, \vec{u_2})$. Since we assumed $\text{Span}(\vec{u_1}, \vec{u_2}) = \text{Span}(\vec{u_1})$, we conclude that $\vec{u_2}$ is also ________________________.

3. (5 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T\left(\binom{x}{y}\right) = \binom{x}{2x+y}$. (It is a good side exercise to check that this is a linear transformation.) In part (a)-(d), I give you a specific set of vectors. For each family, you should compute the linear transformation for several of these vectors, and then describe geometrically what the transformation is doing to this family. For credit you must show your computations **and** give geometric descriptions.

   (a) Consider vectors along the y-axis, so of the form $\binom{0}{y}$.

   (b) Consider vectors along the line $x = 1$, so of the form $\binom{1}{y}$.

   (c) Consider vectors along the line $x = -2$, so of the form $\binom{-2}{y}$.

   (d) Consider vectors along the line $x = \frac{1}{2}$, so of the form $\binom{\frac{1}{2}}{y}$.

   (e) Synthesize your answers in (a)-(d) and describe geometrically what this transformation is doing to the whole plane.

4. (6 points) For each of the following functions $T : \mathbb{R}^2 \to \mathbb{R}^2$, is the function given a linear transformation? If it is, you need to prove both properties. If it is not, you need to give an explicit counterexample.

   (a) $T\left(\binom{x}{y}\right) = \begin{pmatrix} x+3y \\ yx-y \end{pmatrix}$ (b) $T\left(\binom{x}{y}\right) = \begin{pmatrix} x-1 \\ 3y+1 \end{pmatrix}$ (c) $T\left(\binom{x}{y}\right) = \begin{pmatrix} x-3y \\ y\sin^2(x^2) + y\cos^2(x^2) \end{pmatrix}$