Math 215: Linear Algebra

Problem Set 13 : Due December 3

(20 points) Make sure you are familiar with the Academic Honesty policies for this class, as detailed on the syllabus. All work is due on the given day by 3 PM Grinnell Time, or 7 PM if you LaTeX the assignment. **For this assignment, show all of your computational work.**

1. (a) (3 points) Is the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $[T] = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ diagonalizable? If so, find some basis $\alpha = (\vec{u_1}, \vec{u_2})$ so that $[T]_\alpha$ is a diagonal matrix.

   (b) (3 points) Is the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $[T] = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ diagonalizable? If so, find some basis $\alpha = (\vec{u_1}, \vec{u_2})$ so that $[T]_\alpha$ is a diagonal matrix.

2. (4 points) Prove that for all 2×2 matrices A, B we have $\det(A)\det(B) = \det(AB)$ by directly computing using the formulas of matrix multiplication and the determinant.

3. (a) (3 points) Compute the determinant of the transformation R_θ, rotation by an angle of θ. Geometrically, what does this number tell us about this transformation? (Comment particularly on the area distortion of the transformation.)

   (b) (3 points) Compute the determinant of the projection transformation $P_{\vec{w}}$ for any nonzero vector $\vec{w} \in \mathbb{R}^2$. What does this number tell us about this transformation? (Connect your answer to what the projection transformation does geometrically.)

4. (4 points) Let $\mathcal{F}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$, but now define addition on this set to be function composition, so $f + g$ is defined as the function $f \circ g$, and define the zero vector to be the function $z(x) = x$, while scalar multiplication is the standard scalar multiplication defined on this set in Example 4.1.3 in the textbook. Find two vector spaces properties that this set with these two operations and a defined zero vector fail to satisfy. Explain with an explicit example why it fails each of them.

5. (DO NOT TURN IN) Let $V = \mathbb{R}^2$. Define addition of elements as $\binom{x_1}{y_1} + \binom{x_2}{y_2} = \binom{x_1+2x_2}{y_1+3y_2}$ and scalar multiplication as $c\binom{x}{y} = \binom{cx}{cy}$, with the standard zero vector $\binom{0}{0}$. V a not vector space. Find two properties of a vector space which are not satisfied. Carefully explain why each property is not satisfied.

6. (DO NOT TURN IN) Prove that $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is diagonalizable if and only if $b = 0$. (Again, remember how to prove *if and only if* statements.)