Math 215: Linear Algebra

Problem Set 12 : Due November 30

(22 points) Make sure you are familiar with the Academic Honesty policies for this class, as detailed on the syllabus. All work is due on the given day by 3 PM Grinnell Time, or 7 PM if you LaTeX the assignment. **For this assignment, show all of your computational work.**

1. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation with $[T] = \begin{pmatrix} 2 & -1 \\ 5 & 2 \end{pmatrix}$. Let $\vec{u_1} = \binom{1}{3}$ and $\vec{u_2} = \binom{-2}{0}$ with $\alpha = (\vec{u_1}, \vec{u_2})$.
   (a) (2 points) Prove that $\text{Span}(\vec{u_1}, \vec{u_2}) = \mathbb{R}^2$.
   (b) (2 points) Compute $[T(\vec{u_1})]_\alpha$.
   (c) (2 points) Compute $[T(\vec{u_2})]_\alpha$.
   (d) (1 point) Use (b) and (c) to compute $[T]_\alpha$.

2. (2 points) Repeat problem (1) but use Proposition 3.2.6 to determine $[T]_\alpha$ where $\alpha = (\vec{u_1}, \vec{u_2})$.

3. (4 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by $T\left(\binom{x}{y}\right) = \binom{rx}{ry}$ where $r \in \mathbb{R}$ is arbitrary. Prove that for all bases $\alpha = (\vec{u_1}, \vec{u_2})$ of $\mathbb{R}^2$ we have $[T] = [T]_\alpha$.

4. Suppose $[T] = \begin{pmatrix} 35 & 24 \\ -48 & -33 \end{pmatrix}$ and let $\alpha = \left(\binom{-2}{3}, \binom{-3}{4}\right)$ be a basis of $\mathbb{R}^2$.
   (a) (2 points) Compute $T(\vec{v})$ for **four** nonzero vectors $\vec{v} \in \text{Span}(\binom{-2}{3})$.
   (b) (2 points) Compute $T(\vec{v})$ for **four** nonzero vectors $\vec{v} \in \text{Span}(\binom{-3}{4})$.
   (c) (3 points) On one graph, carefully plot the eight vectors $\vec{v}$ from (a) and (b) and the eight vectors $T(\vec{v})$ for those eight vectors. (It is ok to plot them as points in $\mathbb{R}^2$ instead of as vectors.) Geometrically describe what the transformation is doing to these vectors. You should take a picture of your graph and submit it.
   (d) (2 points) Use (c) to find values a and d so that $[T]_\alpha = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Explain briefly your reasoning.

5. (DO NOT TURN IN) Prove that for all 2×2 matrices A we have A is invertible if and only if 0 is not an eigenvalue of A. (Make sure you recall how to prove an *if and only if*.)