**November 27, 2019**

# Gradient, Revisited

Warmup: Given a function of two variables, *f(x,y)* what is the definition of the gradient $\nabla f$? Name a fact you remember about the gradient. Suppose we are given the function

$$f(x, y) = x^2 y - y^3.$$

with gradient:

$$\nabla f = <2xy, x^2 - 3y^2>.$$

For any point (a,b) what kind of object does the gradient give?

# Vector Fields

The gradient is a special case of a more general type of object called a *vector field*. This is one of the word list words for today: function that inputs a point (in $R^2$ or $R^3$) and outputs a vector.

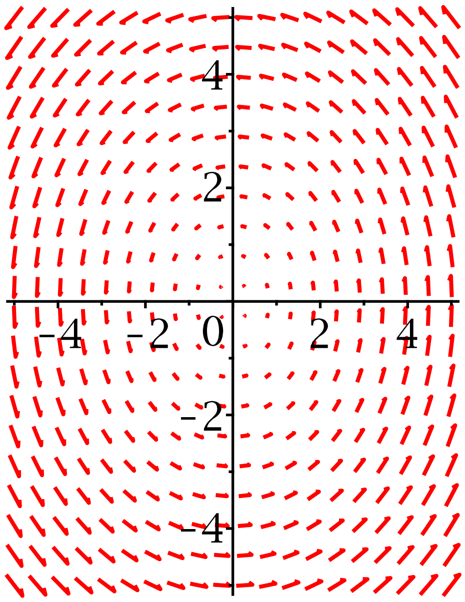
### *An Example*

F(x,y)=<-y,x>. Fill in the chart below with the appropriate vectors.

| $(x, y)$ | $F(x, y)$ |
|---|---|
| $(1, 0)$ | $\langle 0, 1 \rangle$ |
| $(1, 1)$ | |
| $(0, 1)$ | |
| $(-1, 1)$ | |
| $(-1, 0)$ | |
| $(-1, -1)$ | |
| $(0, -1)$ | |
| $(1, -1)$ | |

Decribe what you think will happen. We plot the vector field in the region [-5,5] X [-5,5].

$$fieldplot(vector([-y, x]), x = -5..5, y = -5..5, color = red)$$




Were your predictions above correct? Why or why not? As x gets larger, what happens to the vectors?  As y gets larger what happens to the vectors?

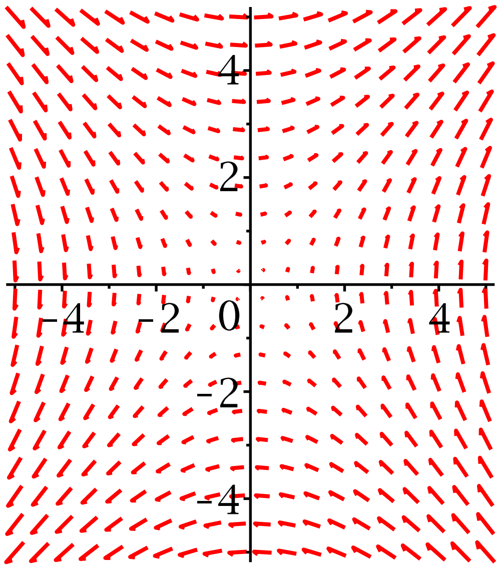
### *Real World Applications*

You may have seen pictures like this before representing wind or water current. Go to *graphical.weather.gov/sectors/northplains.php*

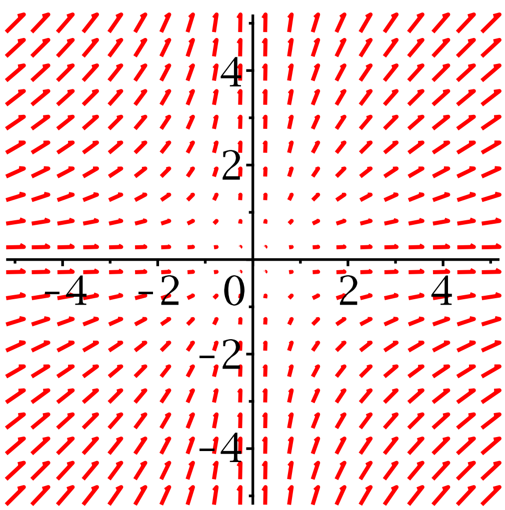
On the left side of that map, mouse over "Wind Speed & Direction" for different hours. Describe what direction and roughly how hard the wind has been blowing in Des Moines today.

### *Some Other Example*

Let **F**(x,y)=$\langle y, x\rangle$.  **Before graphing**, describe in words what you think the vector field will look like.



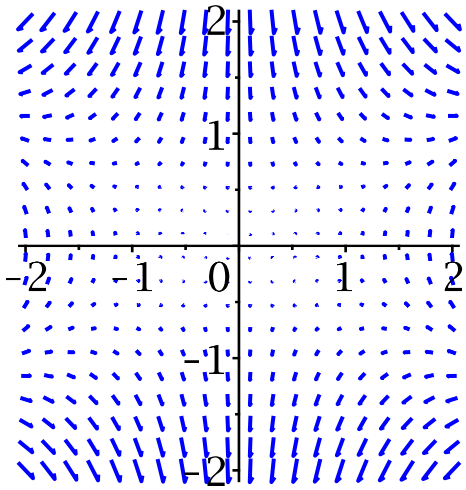

Let's try a more complicated vector field. Let **F**(x,y)=$\langle \ln(1+x^2), \ln(1+y^2)\rangle$ be a vector field. **Before graphing** this, describe roughly what you think this vector field will look like.




## Gradient Vector Fields

There is one special case of vector fields which we worked with early in the semester: How can we interpret the gradient as a vector field? Let's take the function $f(x, y) = x^2 y - y^3$. The *gradient vector field* of this is $\nabla f = \langle 2xy, x^2 - 3y^2\rangle$. Maple has a special command for this.

$gradplot(x^2 \cdot y - y^3, x = -2..2, y = -2..2, color = blue)$



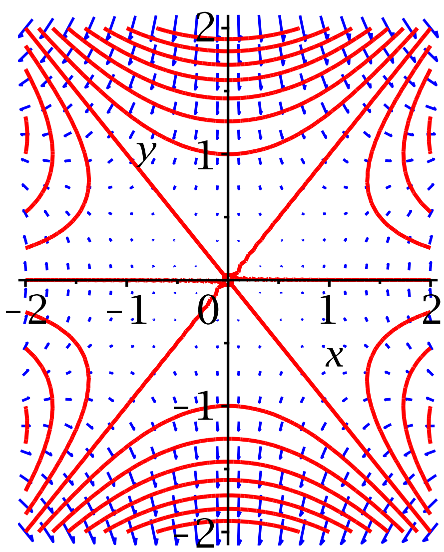

Early in the semester, we made a connection between the gradient at any point and the contour plot of a surface. What was that connection?

Plot together gives us:

$A := gradplot(x^2 \cdot y - y^3, x = -2..2, y = -2..2, color = blue)$

$B := contourplot(x^2 \cdot y - y^3, x = -2..2, y = -2..2, color = red, contours = 15)$

$display(A, B)$

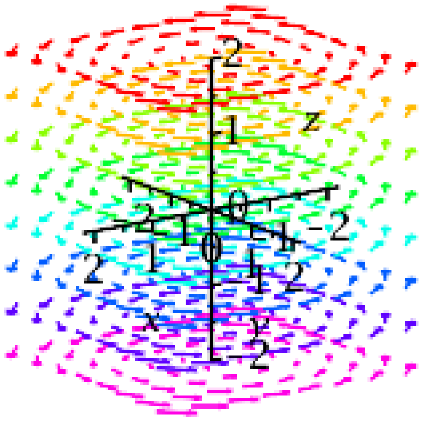



How does this tie in to what you said above about the connection of gradients and contour plots?

## 3D Examples

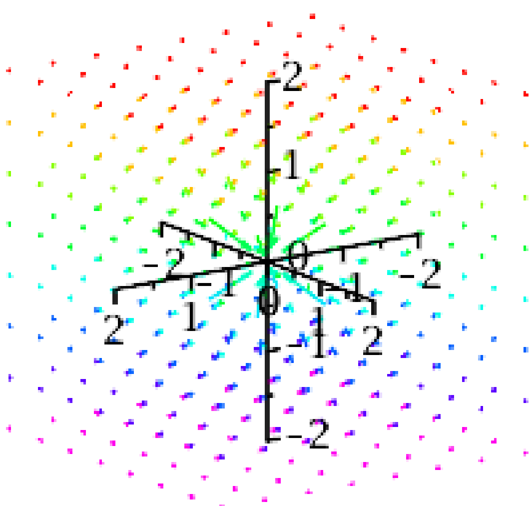
As you can imagine, we can extend the definition of a vector field and a gradient vector field to three dimensions. Name some (i.e. at least 2) 3 dimensional real world situations where these vector fields might be applicable. Let's start with a very simple example again. Suppose $\mathbf{F}(x,y,z)=\langle -y, x, 0\rangle$. Once more, describe what you think this vector field will look like BEFORE we graph it.

$fieldplot3d(vector([-y, x, 0]), x=-2..2, y=-2..2, z=-2..2, axes=normal, arrows = SLIM, shading = zhue)$




### *3d Gradient Vector Fields*

Similarly, we can use the command *gradplot3d* to plot gradient vector fields in 3d. Try out the command on the function $f(x,y,z)=\frac{1}{\sqrt{x^2+y^2+z^2}}$.

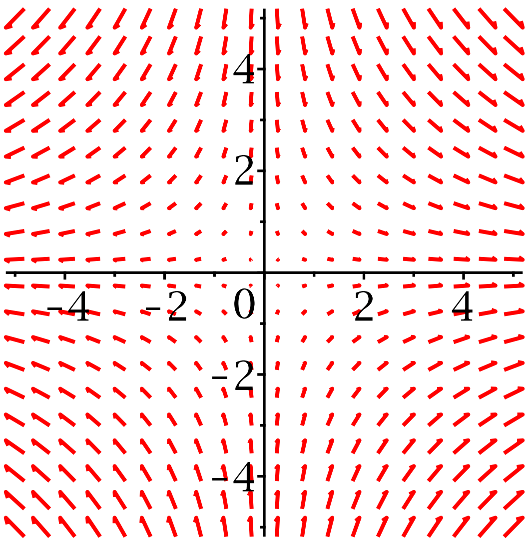



(Comment: we will talk about this function some more next Monday in class..think about a real world situation where the force gets larger as you get closer to the center. )

## Flow Lines

We can interpret vector fields as giving us the velocity at any point (we call these vector fields **velocity fields**). In this setting, we can think about how a particle would move along the vector field (think about the ocean currents we saw above). The path a particle takes through the vector field is called a **flow line**.

First plot the vector field **F**(x,y)= $\langle x, -y \rangle$.



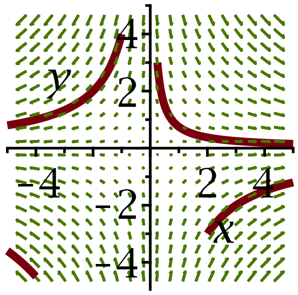

Now take a few sample points and think about what the flow lines will look like. This is comparable to thinking about what would happen to a particle if you dropped it into the vector field at that point. Describe in words what the flow lines will look like. From your descriptions, what do you think the equations of the flow lines is?

### *Plotting Flow Lines*

We use a special package in Maple for plotting flowlines called *Student [VectorCalculus]*. Hit enter on the next line.

$$with(Student[VectorCalculus])$$

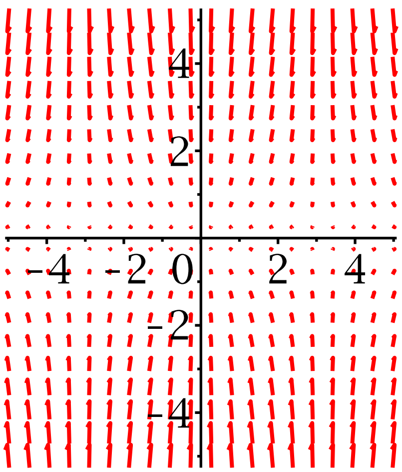
$$FlowLine\left(VectorField(\langle x, -y \rangle), \left[\langle -1, 4 \rangle, \left\langle \frac{1}{4}, 3 \right\rangle, \langle 2, -3 \rangle, \langle -4, -4.5 \rangle\right], scaling = constrained\right)$$




Arrows of the vector field, and the flow line(s) emanating from the given initial point(s)

### *One Last Example*

Let's do one more example of flow lines. Suppose we are given the vector field $F(x, y) = \langle -\sin(x), -2\,y \rangle$. Use the FlowLine command above to explore a few flow lines with x values between -5 and 5. Show your pictures below.




Based on the picture what happens to particles within a radius of about 3 of the origin? What about (0,0), (π,0), and (−π,0). What happens? Why? It turns out that this vector field is the gradient vector field of the function $f(x, y) = \cos(x) - y^2$(such vector fields are called *conservative vector fields*). Recalling everything we know about the connection between gradient vector fields and contour plots, what do you think is so special about the points (0,0) or (π,0) or (−π,0)?

Plot this gradient vector field and the contour plots on one graph, as we did in the "Gradient Vector Field" section. What can you conclude from this picture? Finally, plot the three dimensional surface $f(x, y) = \cos(x) - y^2$ with x between -5 and 5 and y between -2 and 2.

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