Exam 3 Review Solutions

1. (a) Estimate the area under the graph of $f(x) = x^2 - x + 4$ on the interval [1,4] using 6 approximating rectangles and right endpoints.

Here $\Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$. So $x_1 = 1 + \frac{1}{2} = \frac{3}{2}$, $x_2 = \frac{3}{2} + \frac{1}{2} = 2$, $x_3 = \frac{5}{2}$, $x_4 = 3$, $x_5 = \frac{7}{2}$, $x_6 = 4$

$$A \approx \sum_{i=1}^{6} f(x_i)\Delta x = f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\tfrac{1}{2} + f(\tfrac{5}{2})\cdot\tfrac{1}{2} + f(3)\cdot\tfrac{1}{2} + f(\tfrac{7}{2})\tfrac{1}{2} + f(4)\cdot\tfrac{1}{2}$$

$$= \frac{19}{4}\cdot\frac{1}{2} + 6\cdot\frac{1}{2} + \frac{31}{4}\cdot\frac{1}{2} + 10\cdot\frac{1}{2} + \frac{51}{4}\cdot\frac{1}{2} + 16\cdot\frac{1}{2} = \boxed{\frac{229}{8}}$$

(b) Repeat part (a) using left endpoints.

Still $\Delta x = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$, but now $x_0 = a = 1$, $x_1 = 1 + \frac{1}{2} = \frac{3}{2}$, $x_2 = \frac{3}{2} + \frac{1}{2} = 2$, $x_3 = \frac{5}{2}$, $x_4 = 3$, $x_5 = \frac{7}{2}$

$$A \approx \sum_{i=0}^{5} f(x_i)\Delta x = f(1)\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2} + f(\tfrac{5}{2})\cdot\tfrac{1}{2} + f(3)\cdot\tfrac{1}{2} + f(\tfrac{7}{2})\tfrac{1}{2}$$

$$4\cdot\frac{1}{2} + \frac{19}{4}\cdot\frac{1}{2} + 6\cdot\frac{1}{2} + \frac{31}{4}\cdot\frac{1}{2} + 10\cdot\frac{1}{2} + \frac{51}{4}\cdot\frac{1}{2} = \boxed{\frac{181}{8}}$$

2. (a) Express the limit as a definite integral

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(3\left(\frac{3i}{n}\right)^2-4\right)\frac{3}{n}.$$

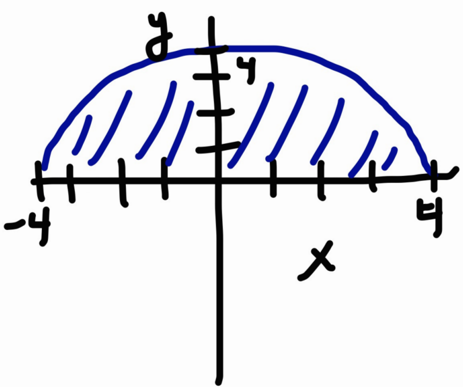
We presume $\Delta x = \frac{3}{n}$. Also $x_1 = \frac{3}{n}$, $x_2 = \frac{3\cdot 2}{n}$, ..., $x_n = \frac{3\cancel{n}}{\cancel{n}} = 3$

So $a=0$, $b=3$. And $f(x) = 3x^2-4$.

So $\boxed{\int_0^3 3x^2-4\,dx}$

(b) Evaluate the integral by interpreting it in terms of area of a familiar shape.

$$\int_{-4}^{4}\sqrt{16-x^2}\,dx$$



The area of this region is ½ area of a circle of radius 4. This gives

$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\cdot 16 = \boxed{8\pi}$

3. Use the definition of the integral (Theorem 4 in Section 5.2) to evaluate $\int_0^3 3 - x^2 dx$ It may help to remember that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

$$\int_0^3 3-x^2dx = \lim_{n\to\infty} \sum_{i=1}^{n} 3-x_i^2\,\Delta x.$$

$\Delta x = \frac{3-0}{n} = \frac{3}{n}$

$x_i = 0 + \frac{3i}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n}\left[3-\left(\frac{3i}{n}\right)^2\right]\cdot\frac{3}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{9}{n}-\frac{27}{n^3}i^2\right) =$$

difference summation rule

constant summation rule

$$= \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{9}{n} - \sum_{i=1}^{n}\frac{27}{n^3}i^2\right) = \lim_{n\to\infty}\left(\frac{9}{\cancel{n}}\cdot\cancel{n} - \frac{27}{n^3}\sum_{i=1}^{n}i^2\right) =$$

$$= \lim_{n\to\infty}\left(9 - \frac{\cancel{27}^{9}}{\cancel{n^3}_{n^2}}\cdot\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}\,2}\right) = \lim_{n\to\infty}\left(9 - \frac{9(2n^2+3n+1)}{2n^2}\right)$$

difference limit law

$$= \lim_{n\to\infty} 9 - \lim_{n\to\infty}\frac{9(2n^2+3n+1)}{2n^2} = 9 - \lim_{n\to\infty}\frac{18+\cancel{\frac{27}{n}}^{0}+\cancel{\frac{9}{n^2}}^{0}}{2}$$

$$= 9-9=0$$

(What does the area of 0 say about this function?)

4. Find the general antiderivatives of the following functions.

(a) $f(x) = x^3 - 2x$

We use the power rule for differentiation in reverse to get $F(x) = \frac{x^4}{4} - x^2$ then

$F'(x) = 4\frac{x^3}{4} - 2x = f(x)$.

Hence $\boxed{F(x) = \frac{x^4}{4} - x^2 + C}$

(b) $f(x) = \sin x$

We are looking for a function whose derivative is
$f(x) = \sin x$.

We know if $F(x) = -\cos x$ then $F'(x) = \sin x$.

Hence $\boxed{F(x) = -\cos x + C}$

(c) $f(x) = \sqrt{x+4}$

$= (x+4)^{1/2}$ We use the power rule for differentiation in reverse to get if

$F(x) = \frac{2}{3}(x+4)^{3/2}$ then $F'(x) = (x+4)^{1/2}$

Hence $\boxed{F(x) = \frac{2}{3}(x+4)^{3/2} + C}$

5. Use logarithmic differentiation to find the derivative of $(\sin x)^{\sqrt{x}}$.

Take the natural log of both sides:

$\ln y = \ln \sin x^{\sqrt{x}} = \sqrt{x} \cdot \ln(\sin x)$ log rules

Implicit Differentiation

$\frac{1}{y} \cdot \frac{dy}{dx} = \sqrt{x} \cdot \frac{1}{\sin x} \cdot \cos x + \ln(\sin x) \cdot \frac{1}{2} x^{-1/2}$ ← product rule

$$\boxed{\frac{dy}{dx} = \left(\sqrt{x} \cdot \frac{\cos x}{\sin x} + \frac{\ln(\sin x)}{2\sqrt{x}}\right) \cdot \underbrace{(\sin x)^{\sqrt{x}}}_{y}}$$

6. Differentiate the following functions.

(a) $f(x) = \ln(x^2 - x)$

Chain Rule

$$f'(x) = \frac{1}{x^2-x} \cdot (2x-1) = \boxed{\frac{2x-1}{x^2-x}}$$

(b) $f(x) = \sin\sqrt{e^{2x} + \ln x}$ Chain Rule

$$f'(x) = \cos\left(\sqrt{e^{2x}+\ln x}\right) \cdot \underbrace{\tfrac{1}{2}\left(e^{2x}+\ln x\right)^{-1/2} \cdot \left(2e^{2x}+\tfrac{1}{x}\right)}_{\text{Derivative of } \sqrt{e^{2x}+\ln x}}$$

(c) $f(x) = \sin^{-1} x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

Inverse Trig

(d) $f(x) = (\sqrt{1 - x^2}) \cdot (\sec^{-1} x)$

$$f'(x) = (\sqrt{1-x^2}) \cdot \frac{1}{x\sqrt{x^2-1}} + (\sec^{-1}x) \cdot \underbrace{\tfrac{1}{2}(1-x^2)^{-1/2} \cdot (-2x)}_{\text{Chain rule}}$$

$h(x) = \sqrt{x}$ $\quad h'(x) = \frac{1}{2}x^{-1/2}$

$g(x) = 1-x^2$ $\quad g'(x) = -2x$

(e) $f(x) = \arccos(x + \tan^{-1} x)$

$h(x) = \arccos(x)$ $\quad h'(x) = \frac{-1}{\sqrt{1-x^2}}$

$g(x) = x + \tan^{-1}(x)$ $\quad g'(x) = 1 + \frac{1}{1+x^2}$

$$f'(x) = h'(g(x)) \cdot g'(x) = \frac{-1}{\sqrt{1-(x+\tan^{-1}(x))^2}} \cdot \left(1 + \frac{1}{1+x^2}\right)$$

(f) $f(x) = \ln|\csc^{-1} x|$

$$f'(x) = \frac{\frac{-1}{x\sqrt{x^2-1}}}{\csc^{-1}x}$$

Log Rule: $\frac{d}{dx}\ln|g(x)| = \frac{g'(x)}{g(x)}$

7. Evaluate the integrals.

(a) $\int x^3 - 2x \; dx$

We use the power rule in reverse to get

$$\boxed{\frac{x^4}{4} - x^2 + C}$$

(b) $\int_0^{\pi} \sin x \; dx$ $= -\cos x \Big|_0^{\pi} = -\cos \pi - (-\cos 0)$

$= -1-(-1) = \boxed{0}$

(c) $\int \frac{1}{x\sqrt{25x^2 - 1}} \; dx$ $\quad 25x^2 = (5x)^2$

substitution

$u = 5x \qquad du = 5dx$

$x = \frac{u}{5} \qquad dx = \frac{1}{5} du$

$$= \int \frac{1}{\frac{u}{5}\sqrt{u^2 - 1}} \frac{1}{5} du = \cancel{5} \cdot \frac{1}{\cancel{5}} \int \frac{1}{u\sqrt{u^2-1}} du =$$

$\sec^{-1} u + C \qquad \boxed{\sec^{-1}(5x) + C}$

(d) $\int_{-1}^{1} \sqrt{x+4} \; dx$ $= \int_{-1}^{1} (x+4)^{1/2} dx = \frac{(x+4)^{3/2}}{3/2} \Big|_{-1}^{1} =$

can do substitution $u = x+4$

$$\frac{2}{3} \cdot \sqrt{(x+4)^3} \Big|_{-1}^{1} = \boxed{\frac{2}{3}\sqrt{5^3} - \frac{2}{3}\sqrt{3^3}}$$

(e) $\int \frac{1}{x(\ln x)^2}\,dx$

substitution
$u = \ln x$
$du = \frac{1}{x}dx$

$= \int \frac{1}{\underbrace{u^2}_{u^{-2}}}\,du = -u^{-1} + C = \boxed{\frac{-1}{\ln x} + C}$

(f) $\int_{-1}^{3} e^{2x}\,dx = \frac{1}{2}e^{2x}\Big|_{-1}^{3} = \boxed{\frac{1}{2}e^{6} - \frac{1}{2}e^{-2}}$

can use substitution
$u = 2x$

(g) $\int_0^2 \sqrt[5]{x^4 - x^2}(10x^3 - 5x)\,dx$

substitution
$u = x^4 - x^2$
$du = (4x^3 - 2x)\,dx$
when $x = 0$, $u = 0$
when $x = 2$, $u = 2^4 - 2^2 = 16 - 4 = 12$

$= \int_0^2 \sqrt[5]{x^4 - x^2} \cdot \frac{5}{2}(4x^3 - 2x)\,dx$

$= \int_0^{12} (u)^{1/5} \cdot \frac{5}{2}\,du = \frac{5}{2}\,\frac{u^{6/5}}{6/5}\Big|_0^{12} = \boxed{\frac{25}{12}(12)^{6/5}} - 0$

(h) $\int \frac{x}{3x^2 - 5}\,dx$

substitution
$u = 3x^2 - 5$
$du = 6x\,dx$
$\frac{1}{6}du = x\,dx$

$= \int \frac{1}{6} \cdot \frac{1}{u}\,du = \frac{1}{6}\ln|u| + C = \boxed{\frac{1}{6}\ln|3x^2 - 5| + C}$

8. Solve for x in each of the following equations.

(a) $\ln(x^2 + 1) - 3\ln(x) = \ln(2)$.

Apply log rules first

$\ln(x^2+1) - \ln(x^3) = \ln(2)$ or $\ln\left(\frac{x^2+1}{x^3}\right) = \ln(2)$

So $\frac{x^2+1}{x^3} = 2$ $2x^3 = x^2+1$

$2x^3 - x^2 - 1 = 0$

$x = 1$

long division to find other roots

$$\begin{array}{r} 2x^2+x+1 \\ x-1 \,\big)\, 2x^3 - x^2 \quad - 1 \\ -(2x^3 - 2x^2) \\ \hline x^2 \\ -(x^2 - x) \\ \hline x - 1 \\ -(x-1) \\ \hline 0 \end{array}$$

So $2x^3 - x^2 - 1 = (x-1)(2x^2+x+1)$

Other roots are

$\frac{-1 \pm \sqrt{1-4\cdot 2\cdot 1}}{2\cdot 2}$ ← not real #s

So $\boxed{x=1}$ only solution.

(b) $e^{2x+1} = 9e^{1-x}$

Divide by e^{1-x}

$\frac{e^{2x+1}}{e^{1-x}} = 9$

$e^{2x+1-(1-x)} = 9$

$e^{3x} = 9$

$\ln e^{3x} = \ln 9$

$3x = \ln 9$ $\boxed{x = \frac{\ln 9}{3}}$

$2e^{1-x^2}$

9. Find the equation of the tangent line to the graph $f(x) = \cancel{2e^{1x}}$ at the point $(1, 2)$.

We need a point on the line and the slope of $f(x)$ at that point.

Point is $(1,2)$

Slope: $f'(x) = 2e^{1-x^2} \cdot (-2x)$

↑ chain rule

$f'(1) = 2e^{1-1^2}(-2 \cdot 1) = 2 \cdot e^0 (-2) = -4$

Apply Point-Slope Formula

$y - 2 = (-4)(x - 1)$

$y = -4x + 4 + 2$

$\boxed{y = -4x + 6}$

10. Compute the derivative of the function

$$f(x) = \int_1^x \cot t \; dt$$

By the Fundamental Theorem, Part I, this derivative is $\boxed{f'(x) = \cot x}$

11. Find the integral $\displaystyle\int_1^{\sqrt{3}} \frac{4}{1+x^2} \, dx.$

$= 4\int_1^{\sqrt{3}} \frac{1}{1+x^2} dx = 4\tan^{-1} x \Big]_1^{\sqrt{3}} = 4\tan^{-1}\sqrt{3} - 4\tan^{-1} 1 = \boxed{4 \cdot \frac{\pi}{3} - 4 \cdot \frac{\pi}{4}}$