Exam 2 Review Solutions

(1) Differentiate the following functions.

(a) $f(x) = \tan(x + \tan x)$ Chain Rule

$h(x) = \tan x \qquad h'(x) = \sec^2 x$

$g(x) = x + \tan x \qquad g'(x) = 1 + \sec^2 x$

$f'(x) = h'(g(x)) \cdot g'(x) = \underbrace{\sec^2(x + \tan x)}_{h'(g(x))} \cdot \underbrace{(1 + \sec^2 x)}_{g'(x)}$

(b) $g(r) = \sqrt{9 + r + \sin 3r}$ Chain Rule

$h(r) = \sqrt{r} = r^{1/2} \qquad h'(r) = \frac{1}{2} r^{-1/2}$

$f(r) = 9 + r + \sin 3r \qquad g'(r) = 1 + \cos(3r) \cdot 3$ Chain rule here too

$g'(r) = h'(f(r)) \cdot g'(r) = \underbrace{\frac{1}{2}(9 + x + \sin(3x))^{-1/2}}_{h'(f(r))} \cdot \underbrace{(1 + \cos(3x) \cdot 3)}_{f'(r)}$

Chain Rule again!

(c) $x(t) = (\cot(t^2))^5$

$f(t) = t^5 \qquad f'(t) = 5t^4$

$g(t) = \cot(t) \qquad g'(t) = -\csc^2(t)$

$h(t) = t^2 \qquad h'(t) = 2t$

$x'(t) = f'(g(h(t))) \cdot g'(h(t)) \cdot h'(t) = \underbrace{5(\cot(t^2))^4}_{f'(g(h(t)))} \cdot \underbrace{(-\csc^2(t^2))}_{g'(h(t))} \underbrace{2t}_{h'(t)}$

(2) Find $f'(x)$ by implicit differentiation for $4xy - \tan(y) = 3x^2 + \sin(x)$.

Taking a derivative of both sides

$$\underbrace{4x\frac{dy}{dx} + 4\cdot y}_{\text{productrule}} - \underbrace{\sec^2(y)\cdot\frac{dy}{dx}}_{\text{chain rule}} = 6x + \cos(x)$$

Factoring $\frac{dy}{dx}$ out

$$\frac{dy}{dx}(4x - \sec^2(y)) + 4y = 6x + \cos(x)$$

move 4y to the other side

$$\frac{dy}{dx}(4x - \sec^2(y)) = 6x + \cos(x) - 4y$$

Divide by $4x - \sec^2(y)$ on bothsides →

$$\boxed{\frac{dy}{dx} = \frac{6x + \cos(x) - 4y}{4x - \sec^2(y)}}$$

(3) Verify that the function $f(x) = 4 + \sqrt{x-1}$ on the interval [1,5] satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

This function is continuous on $[1,5]$ since $\sqrt{x-1}$ is defined for $x \geq 1$.
The function is differentiable on $(1,5)$ since we can compute the derivative $f'(x) = \frac{1}{2}(x-1)^{-1/2}\cdot 1 = \frac{1}{2\sqrt{x-1}}$ (Notice it is not differentiable at $x=1$.)

Now consider $\frac{f(5)-f(1)}{5-1} = \frac{(4+\sqrt{4}) - 4 + \sqrt{1-1}}{4} = \frac{6-4}{4} = \frac{1}{2}$

So we find c so that $f'(c) = \frac{1}{2}$

$$\frac{1}{2\sqrt{c-1}} = \frac{1}{2} \text{ if } \frac{1}{\sqrt{c-1}} = 1 \text{ or } \sqrt{c-1} = 1 \text{ or } c-1 = 1 \text{ so } c=2$$

(4) Use the following table of values to calculate the derivative of the given functions at $x = 2$.

x	$g(x)$	$h(x)$	$g'(x)$	$h'(x)$
2	5	4	-3	9
4	3	2	2	3

(a) $f(x) = \dfrac{g(x)}{h(x)}$ start with quotient rule

$$f'(x) = \frac{h(x)\cdot g'(x) - g(x)\cdot h'(x)}{h^2(x)} \qquad f'(2) = \frac{h(2)\cdot g'(2) - g(2)h'(2)}{h^2(2)} = \frac{4\cdot(-3) - 5\cdot 9}{16} = \boxed{\frac{-57}{16}}$$

(b) $f(x) = g(h(x))$ Start with chain rule

$$f'(x) = g'(h(x))\cdot h'(x)$$

$$f'(2) = g'(h(2))\cdot h'(2) = g'(4)\cdot 9 = (-2)\cdot 9 = -18$$

(5) What is the equation for the tangent line of $x^2 + (3y)^2 = 13$ at the point (2,1)?

For the tangent line we need the slope at $x=2, y=1$.
We use implicit differentiation

$$2x + 2\cdot(3y)3\frac{dy}{dx} = 0$$

$$18y\frac{dy}{dx} = -2x$$

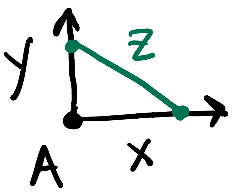
$$\frac{dy}{dx} = \frac{-2x}{18y} = \frac{-x}{9y}$$

at (2,1) this is $\dfrac{-2}{9\cdot 1} = \dfrac{-2}{9}$

Then using point-slope formula

$$\boxed{y - 1 = \frac{-2}{9}(x-2)}$$

(6) A girl starts at a point A and runs east at a rate of 10 feet/second. One minute later, another girl starts at A and runs north at a rate of 8 feet/second. At what rate is the distance between them changing 1 minute after the second girl starts?



known rates: $\frac{dx}{dt} = 10 \text{ ft/sec}$

$\frac{dy}{dt} = 8 \text{ ft/sec}$

unknown rate: $\frac{dz}{dt}$

@ 2 minutes

$x = 120 \cdot 10 = 1200$

$y = 60 \cdot 8 = 480$

equation: $x^2 + y^2 = z^2$

$z = \sqrt{x^2 + y^2}$

$z \approx 1292.4$

differentiate: $\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2}z\frac{dz}{dt}$

$$\frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} = \frac{dz}{dt}$$

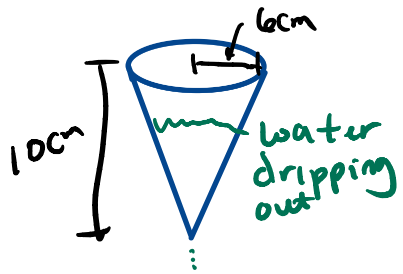
plug in values. $\frac{dz}{dt} = \frac{1200 \cdot 10 + 480 \cdot 8}{1292.4} \approx \boxed{12.3 \text{ ft/sec}}$

(7) Let $R(x)$ be a function that measures a company's revenue R from car sales (in thousands of dollars) in terms of advertising expenditures, x (also in thousands of dollars). Suppose the company is spending \$100,000 on advertising right now. If $R'(100) = -10$ should the company spend more or less on advertising to increase revenue? Why?

Since the derivative is negative, this means the function is decreasing. This means that as x increases, R decreases and as x decreases, R increases. Thus the company should decrease spending to increase revenue.

(8) A cone-shaped coffee filter of radius 6 cm and depth 10 cm contains water, which drips out through a hole at the bottom so the volume of the water in the filter decreases at a rate of 1.5 cm^3/sec. How fast is the water level falling when the depth is 8 cm? (Hint: the volume of a cone is $\frac{1}{3}\pi r^2 h$.)

This problem requires you to know that for a cone, the ratio of the height to radius is the same as the water decreases



so $\frac{h}{r} = \frac{10}{6}$ or $r = \frac{6h}{10} = \frac{3h}{5}$

Know rate $\frac{dV}{dt} = -1.5 cm^3/sec$ unknown rate $\frac{dh}{dt}$

equation relating the variables in the rate $V = \frac{1}{3}\pi r^2 h = \frac{1}{\cancel{3}}\pi\left(\frac{\cancel{3}h}{5}\right)^2 \cdot h = \frac{3\pi}{25}h^3$

Take a derivative with respect to time $\frac{dV}{dt} = \frac{9\pi}{25}h^2 \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{25}{9\pi h^2} \cdot \frac{dV}{dt}$

plug in @ $h = 8$

$\frac{dh}{dt} = \frac{25}{9\pi \cdot 8^2} \cdot (-1.5) = \boxed{\frac{-25}{384\pi} \text{ cm/sec}}$

$-\frac{3}{2}$

(9) Let $f(x) = \cos x + \frac{\sqrt{3}}{2}x$ on the interval $0 \le x \le 2\pi$.

(a) Find the critical number(s) of the function.

We find where $f'(x) = 0$ or undefined.

$f'(x) = -\sin x + \frac{\sqrt{3}}{2}$ ← never undefined.

$0 = -\sin x + \frac{\sqrt{3}}{2}$

$\sin x = \frac{\sqrt{3}}{2}$

$\boxed{x = \pi/3 \text{ and } x = \frac{2\pi}{3}}$

$$\begin{array}{c|c|c|c|} & + & - - - & + + + + + + + \\ 0 & & \frac{\pi}{3} \quad \frac{2\pi}{3} & & 2\pi \end{array} \quad f'$$

(b) Find the intervals on which f is increasing or decreasing.

By increasing/decreasing test
increasing $(0, \frac{\pi}{3}) \cup (\frac{2\pi}{3}, 2\pi)$ and decreasing $(\frac{\pi}{3}, \frac{2\pi}{3})$

(c) Use the first derivative test to find local maximum and minimum values of f.

At $x = \frac{\pi}{3}$ the derivative goes from positive to negative so this is a maximum.

$$f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\left(\frac{\pi}{3}\right) = \boxed{\frac{1}{2} + \frac{\sqrt{3}}{6}\pi}$$

At $x = \frac{2\pi}{3}$ the derivative goes from negative to positive so this is a minimum.

$$f\left(\frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) + \frac{\sqrt{3}}{2}\left(\frac{2\pi}{3}\right) = \boxed{-\frac{1}{2} + \frac{\sqrt{3}}{3}\pi}$$

(10) Let $f(x) = -x^4 + 2x^3 + x^2 - 2$.

(a) Find the critical number(s) of the function.

$$f'(x) = -4x^3 + 6x^2 + 2x$$
$$= -2x(2x^2 - 3x - 1)$$

$$2x^2 - 3x - 1 = 0$$

Critical #s at $x = 0$ or $\dfrac{3 \pm \sqrt{9 + 4 \cdot 2}}{2 \cdot 2} = \dfrac{3 \pm \sqrt{17}}{4}$

(b) Using the second derivative test, find the local maximum and minimum values of f.

$$f''(x) = -12x^2 + 12x + 2$$

$f''(0) = 2 \longrightarrow +$ minimum

$f''\left(\frac{3+\sqrt{17}}{4}\right) = -12\left(\frac{3+\sqrt{17}}{4}\right)^2 + 12\left(\frac{3+\sqrt{17}}{4}\right) + 2 \approx -14.685$ since negative maximum

$f\left(\frac{3-\sqrt{17}}{4}\right) = -12\left(\frac{3-\sqrt{17}}{4}\right)^2 + 12\left(\frac{3-\sqrt{17}}{4}\right) + 2 \approx -2.315$ since negative maximum

(c) Find the interval(s) where the function is concave upward and concave downward.

We need to find where $f''(x) > 0$ and $f''(x) < 0$. First we find the zeros of $f''(x)$ using the quadratic formula.

$$x = \frac{-12 \pm \sqrt{144 - 4(-12)2}}{-24} = \frac{-12 \pm \sqrt{144 + 96}}{-24} = \frac{-12 \pm \sqrt{240}}{-24} = \frac{1}{2} \pm \frac{4\sqrt{15}}{24} = \frac{1}{2} \pm \frac{\sqrt{15}}{6}$$

$f''(0) = 2$
$f''(2) = -12 \cdot 4 + 12 \cdot 2 + 2 = -22$
$f''(-1) = -12 - 12 + 2 = -22$

f'': $- - - - \mid + + + + \mid - - - -$

$-0.1455 \approx \frac{1}{2} - \frac{\sqrt{15}}{6}$ $\qquad$ $\frac{1}{2} + \frac{\sqrt{15}}{6} \approx 1.1455$

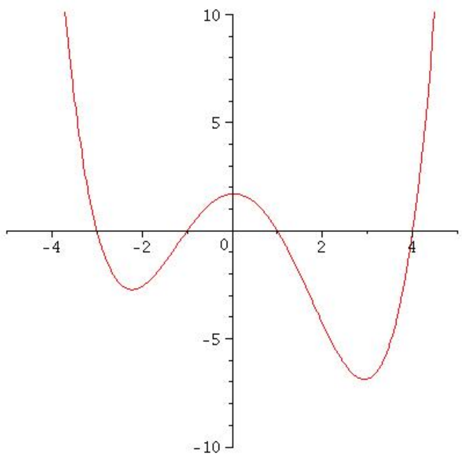
So concave down $\left(-\infty, \frac{1}{2} - \frac{\sqrt{15}}{6}\right) \cup \left(\frac{1}{2} + \frac{\sqrt{15}}{6}, \infty\right)$

concave up $\left(\frac{1}{2} - \frac{\sqrt{15}}{6}, \frac{1}{2} + \frac{\sqrt{15}}{6}\right)$

(d) Find the inflection point(s).

The inflection points are at
$x = \frac{1}{2} - \frac{\sqrt{15}}{6}$ and $x = \frac{1}{2} + \frac{\sqrt{15}}{6}$

(11) The graph of the **derivative** f' of a function f is shown below.



(a) On what intervals is f increasing or decreasing?

f increases when the derivative is positive so $(-\infty,-3)\cup(-1,1)\cup(4,\infty)$

f decreases when derivative is negative so $(-3,-1)\cup(1,4)$

(b) At what values of x does f have a local maximum or minimum?

where the derivative is 0

$\boxed{x=-3,\ -1,\ 1,\ \text{and } 4}$

(12) (a) Find the x and y intercepts and any asymptotes of the function

$$f(x)=\frac{x^3+3x^2-x-3}{x^2+1}$$

$x^2(x+3)-(x+3)$

$(x^2-1)(x+3)$

x intercepts

$\boxed{x=\pm 1,-3}$

y intercepts

$f(0)=-3$

Degree of numerator is one more than degree of denominator so slant asymptote

No vertical asymptotes since the denominator is never 0.

$$\begin{array}{r} x+3 \\ x^2+1\,\overline{)\,x^3+3x^2-x-3} \\ -(x^3 \qquad\quad +x) \\ \hline 3x^2-2x-3 \\ -3x^2-3x \\ \hline x-3 \end{array}$$

$\boxed{\text{Slant of } y=x+3}$

(b) Is $f(x)=x^4-\cos(x)$ even, odd, or neither?

$f(-x)=(-x)^4-\cos(-x)=x^4-\cos(x)$

cos x is an even function

$-f(x)=-x^4+\cos x$

Since $f(-x)=f(x)$, this function is even.

(13) Find the following limits.

(a) $\lim_{x\to\infty} \frac{3x^2-x+4}{5x-7x^2}$ Multiply top and bottom by $1/x^2$:

$$= \lim_{x\to\infty} \frac{(3x^2-x+4)\cdot 1/x^2}{(5x-7x^2)\cdot 1/x^2} = \lim_{x\to\infty} \frac{3-\frac{1}{x}+\frac{4}{x^2}}{\frac{5}{x}-7} \overset{\text{quotient limit rule}}{=} \frac{\lim_{x\to\infty}\left(3-\frac{1}{x}-\frac{4}{x^2}\right)}{\lim_{x\to\infty}\left(\frac{5}{x}-7\right)}$$

Sum or difference rule

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{4}{x^2}}{\lim_{x\to\infty}\frac{5}{x} - \lim_{x\to\infty} 7} \overset{\text{constant multiple rule}}{=} \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{1}{x} - 4\lim_{x\to\infty}\frac{1}{x^2}}{5\lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty} 7} \overset{\text{super rule!}}{=} \frac{3-0-0}{0-7} = \boxed{\frac{-3}{7}}$$

constant rule

(b) $\lim_{x\to\infty} \frac{x^2}{x^3-3x^2+x-1}$

Now multiply by $\frac{1}{x^3}$ to get:

$$\lim_{x\to\infty} \frac{x^2\cdot\frac{1}{x^3}}{(x^3-3x^2+x-1)\cdot 1/x^3} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1-\frac{3}{x}+\frac{1}{x^2}-\frac{1}{x^3}} \overset{\text{quotient limit rule}}{=} \frac{\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty}\left(1-\frac{3}{x}+\frac{1}{x^2}-\frac{1}{x^3}\right)}$$

Sum or difference rule

$$= \frac{\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty}\frac{3}{x} + \lim_{x\to\infty}\frac{1}{x^2} - \lim_{x\to\infty}\frac{1}{x^3}} = \frac{\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty} 1 - 3\lim_{x\to\infty}\frac{1}{x} + \lim_{x\to\infty}\frac{1}{x^2} - \lim_{x\to\infty}\frac{1}{x^3}} \overset{\text{constant rule}}{=} \frac{0}{1-0+0-0} = \boxed{0}$$

constant multiple rule

$\lim_{x\to\infty}\frac{1}{x^k}=0$ super rule!

(c) $\lim_{x\to-\infty} \frac{4x^3+2x^2-1}{2x^2-1}$

Here multiply by $\frac{1}{x^2}$

$$\lim_{x\to-\infty} \frac{4x+2-\frac{1}{x}}{2-\frac{1}{x}} \overset{\text{quotient limit law}}{=} \frac{\lim_{x\to-\infty} 4x+2-\frac{1}{x}}{\lim_{x\to-\infty} 2-\frac{1}{x}} \overset{\text{sum/difference limit law}}{=} \frac{\lim_{x\to-\infty} 4x + \lim_{x\to-\infty} 2 - \lim_{x\to-\infty}\frac{1}{x}}{\lim_{x\to-\infty} 2 - \lim_{x\to\infty}\frac{1}{x}} \overset{\text{constant \& super rule!}}{=}$$

$$\frac{\lim_{x\to-\infty} 4x + 2}{2}$$

Since the denominator approaches 2 and the numerator grows negatively without bound, the limit is $\boxed{-\infty}$

(d) $\lim_{x\to\infty} \sqrt{5x^4-3x+1} - \sqrt{5x^4-2x^2+x+4}$ multiply by the conjugate

multiply by $1/x^2$ top & bottom

$$= \lim_{x\to\infty} \frac{\cancel{5x^4}-3x+1-\cancel{5x^4}+2x^2-x-4}{\sqrt{5x^4-3x+1}+\sqrt{5x^4-2x^2+x+4}} = \lim_{x\to\infty} \frac{(2x^2-4x-3)\cdot 1/x^2}{\left(\sqrt{5x^4-3x+1}+\sqrt{5x^4-2x^2+x+4}\right)\cdot\frac{1}{\sqrt{x^4}}}$$

Quotient limit law; Sum & difference limit law

$$= \lim_{x\to\infty} \frac{2-4/x-3/x^2}{\sqrt{5-3/x^3+1/x^4}+\sqrt{5-\frac{2}{x^2}+\frac{1}{x^3}+\frac{4}{x^4}}} = \frac{\lim_{x\to\infty} 2-4/x-3/x^2}{\lim_{x\to\infty}\sqrt{5-3/x^3+1/x^4}+\sqrt{5-\frac{2}{x^2}+\frac{1}{x^3}+\frac{4}{x^4}}} = \frac{\lim_{x\to\infty} 2-\lim_{x\to\infty} 4/x-\lim_{x\to\infty} 3/x^2}{\sqrt{\lim_{x\to\infty} 5-3/x^3+1/x^4}+\sqrt{\lim_{x\to\infty} 5-\frac{2}{x^2}+\frac{1}{x^3}+\frac{4}{x^4}}} =$$

limit law (11) + sum rule

$$= \frac{\lim_{x\to\infty} 2-\lim_{x\to\infty} 4/x-\lim_{x\to\infty} 3/x^2}{\sqrt{\lim_{x\to\infty} 5-\lim_{x\to\infty} 3/x^3+\lim_{x\to\infty} 1/x^4}+\sqrt{\lim_{x\to\infty} 5-\lim_{x\to\infty} \frac{2}{x^2}+\lim_{x\to\infty}\frac{1}{x^3}+\lim_{x\to\infty}\frac{4}{x^4}}} = \frac{\lim_{x\to\infty} 2-4\lim_{x\to\infty} 1/x-3\lim_{x\to\infty} 1/x^2}{\sqrt{\lim_{x\to\infty} 5-3\lim_{x\to\infty} 1/x^3+\lim_{x\to\infty} 1/x^4}+\sqrt{\lim_{x\to\infty} 5-2\lim_{x\to\infty}\frac{1}{x^2}+\lim_{x\to\infty}\frac{1}{x^3}+4\lim_{x\to\infty}\frac{1}{x^4}}}$$

constant multiple rule

constant rule & super rule!

$$= \frac{2}{\sqrt{5}+\sqrt{5}} = \frac{2}{2\sqrt{5}} = \boxed{\frac{1}{\sqrt{5}}}$$

(14) A cylindrical container, open at the top and of capacity 24π cubic inches is to be manufactured. If the cost of the material used for the bottom of the container is 6 cents per square inch, and the cost of the material used for the curved part is 2 cents per square inch, find the dimension which will minimize the cost. (Hint: The bottom of the cylinder is a circle and the curved part is really a rectangle--visualize cutting open a can and unfolding the curved part -- with height h and length the circumference of the bottom.)

(1)

(2) The objective function is cost: $C = 6\cdot\underbrace{\pi r^2} + 2\cdot\underbrace{2\pi r h}$ see above.

(3) We know $V = 24\cancel{\pi} = \cancel{\pi} r^2 h$ so $h = 24/r^2$

Plugging in we get $C(r) = 6\pi r^2 + 4\pi r\,(24/r^2) = 6\pi r^2 + \frac{96\pi}{r}$ ← minimize this objective function

(4) $C'(r) = 12\pi r - \frac{96\pi}{r^2} = 0$ if $12\cancel{\pi} r = \frac{96\cancel{\pi}}{r^2}$

$12r^3 = 96$

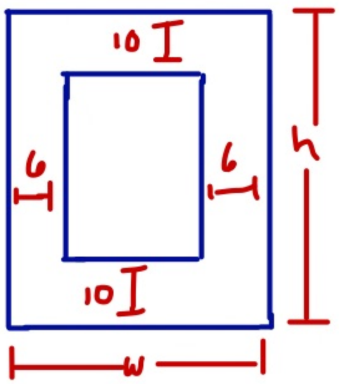
$r^3 = 8$ or $\boxed{r=2}$

Confirm min: $C''(r) = 12\pi + \frac{96}{3}\pi\cdot\frac{1}{r^3}$ so $C''(2) > 0$ hence min.

(5) Answer the question: When $r=2$, $h = 24/2^2 = 6$

So dimensions are $\boxed{\text{6 inches high × 2 inch radius}}$

(15) A poster of area 6000cm^2 has blank margins of width 10 cm on both the top and bottom and 6cm on each side. Find the dimensions that maximize the printed area.

① 

② $A_p = (w-12)(h-20)$

③ We know $A = 6000 = w \cdot h$ so $w = 6000/h$

Plugging in we get $A_p(h) = \left(6000/h - 12\right)(h-20)$

$$A_p(h) = 6000 - \frac{120000}{h} - 12h + 240$$

maximize this objective function

④ $A_p'(h) = \dfrac{120000}{h^2} - 12$. Find where this is zero.

$$\frac{120000}{h^2} = 12$$

or

$$h^2 = 10000$$

$$h = \pm 100$$ (−100 doesn't make sense)

Confirm maximum

$A_p''(h) = \dfrac{-120000 \cdot 2}{h^3}$ so $A_p''(100) < 0$ so max.

⑤ Answer the question.

When $h = 100$, $w = 6000/h = 6000/100 = 60$

So $\boxed{100\text{ cm} \times 60\text{ cm}}$