Math 218: Combinatorics

Homework 9 : Due October 15

Note: one of the 7 problems is on the 2nd page.

1. A chocolate bar consists of n squares arranged in a rectangular pattern. You split the bar into small squares, always breaking along the lines between the squares. Use induction to prove that no matter how you break the bar, it takes $n-1$ breaks to split it into the n smaller squares.

   Comment: Chocolate bars are not necessarily one long line of rectangles. When $n = 6$ the bar could consist of 6 small squares in a row, or it could consist of two rows of 3 squares each. Here is a picture of a chocolate bar, and some physics on why they typically break at the seams: http://physics.stackexchange.com/questions/238202/why-do-chocolate-bars-usually-break-at-the-cleavages

2. Suppose you want to mail a letter but all you have are five and six cent stamps. Determine (with proof) all possible postage values you can create with those five and six cent stamps. (For example, you could have an 11 cent postage by putting a 5 cent stamp and a 6 cent stamp on the letter.)

3. Determine whether each of the following relations is reflexive, symmetric, and transitive (you should check each individual property, not all three at once). If a certain property fails, you should give a specific counterexample. (This problem will be worth 6 points.)

   a. $S = \mathbb{Z}$ where $a \sim b$ means $a - b \neq 1$.

   b. $S = \mathbb{Z}$ where $a \sim b$ means that both a and b are even.

   c. $S = \mathbb{Z}$ where $a \sim b$ means $a \mid b$.

4. Let $S = \mathbb{Z} \times \mathbb{Z}$, the set of ordered pairs of integers, and define $(a, b) \sim (c, d)$ if $ad = bc$. Prove that this is an equivalence relation. What are the equivalence classes?

5. Define a relation on the set of all lists of n distinct integers chosen from $[n]$, by saying two lists are related if they have the same elements (though perhaps in a different order) in the first k places, and the same elements (though perhaps in a different order) in the last $n - k$ places. Show this relation is an equivalence relation.

6. Consider now all lists of n distinct integers chosen from $[n]$ by saying two lists are related if they have the same elements in order, except perhaps shifted. Show this is an equivalence relation. What are the equivalence classes and how does this connect to a counting problem from earlier in the semester?

7. A friend tries to convince you that the reflexive property is redundant in the definition of an equivalence relation because they claim that symmetry and transitivity imply it. Here is the argument they propose:

   "If $a \sim b$, then $b \sim a$ by symmetry, so $a \sim a$ by transitivity. This gives the reflexive property."
   Now you know that their argument must be wrong because one of the examples in Problem 3 is symmetric and transitive but not reflexive. Pinpoint the error in your friend's argument. Be as explicit and descriptive as you can.