Jacobian Varieties of Hurwitz Curves

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Introduction

A curve with the property that the size of its automorphism group attains the maximum bound for its genus is called a *Hurwitz curve*. In certain cases these curves have automorphism group PSL$(2, q)$, the projective special linear group of 2×2 matrices with entries in $\mathbb{F}_q$.

Theorem (MacBeath, [2])

PSL$(2, q)$ is the automorphism group of a Hurwitz curve if and only if
(1) $q = 7$ or
(2) q is a prime and congruent to $\pm 1 \bmod 7$ or
(3) $q = p^3$ for a prime $p \equiv \pm 2$ or $\pm 3 \bmod 7$.

Let X be such a curve with automorphism group G =PSL$(2, q)$, g the genus of X, and JX its Jacobian variety. We study factorizations of JX using representation theory and techniques from [3]. Since

$$\mathbb{Q}G \cong \bigoplus_i M_{n_i}(\Delta_i) \tag{1}$$

where $M_{n_i}(\Delta_i)$ is $n_i \times n_i$ matrices with coefficients in a division ring Δ_i, our decomposition is:

$$JX \sim \bigoplus_i (e_i(JX))^{n_i} \tag{2}$$

where e_i are certain idempotents in End$(JX) \otimes_{\mathbb{Z}} \mathbb{Q}$. Also,

$$\dim(e_i(JX)) = \frac{1}{2}\langle \chi, \varphi_i \rangle \tag{3}$$

where $\langle \chi, \varphi_i \rangle$ denotes the inner product of χ, a special character we define below, with φ_i, the ith irreducible $\mathbb{Q}$-character, labeled according to the decomposition in (1).

Properties of PSL$(2, q)$

The conjugacy classes of PSL$(2, q)$ are generated by the identity element e_G, elements we call c and d, and powers of elements called a and b where $o(a) = \frac{q-1}{2}$, $o(b) = \frac{q+1}{2}$, and $o(c) = o(d) = q$. Define ε to be a primitive $(q-1)$-th root of unity and δ a primitive $(q+1)$-th root of unity with $\varepsilon_{kn} = \varepsilon^{2kn} + \varepsilon^{-2kn}$ and $\delta_{tm} = -(\delta^{2tm} + \delta^{-2tm})$.

Table 1 : Character Table of PSL$(2, q)$ when $q \equiv 1 \bmod 4$.

	$[e_G]$	$[a^n]$	$[b^m]$	$[c]$	$[d]$
1_G	1	1	1	1	1
λ	q	1	-1	0	0
μ_k	$q+1$	ε_{kn}	0	1	1
θ_t	$q-1$	0	δ_{tm}	-1	-1
χ_1	$\frac{q+1}{2}$	$(-1)^n$	0	$\frac{1+\sqrt{q}}{2}$	$\frac{1-\sqrt{q}}{2}$
χ_2	$\frac{q+1}{2}$	$(-1)^n$	0	$\frac{1-\sqrt{q}}{2}$	$\frac{1+\sqrt{q}}{2}$

Here $1 \le m, n, t \le \frac{q-1}{4}$ and $1 \le k \le \frac{q-5}{4}$. There is a similar table for $q \equiv -1 \bmod 4$. The irreducible $\mathbb{Q}$-characters are 1_G, λ, $\chi_1 + \chi_2$, combinations of the μ_k for $d \mid \frac{q-1}{2}$, and combinations of the θ_t for $d \mid \frac{q+1}{2}$.

The automorphism groups of a Hurwitz curves are finite quotients of the $(2, 3, 7)$ triangle group so monodromy of the covering $X \to X/G$ (which we will need to compute χ) consists of one element each in G of order 2, 3, and 7. The following result helps us determine the monodromy.

Proposition

When G =PSL$(2, q)$ for q odd, $q > 27$, and q satisfying one of (2) or (3) in the Theorem above, then G has three distinct conjugacy classes of elements of order 7, and one each of elements of order 2 and 3.

Calculation of χ

Let g_2, g_3, and g_7 be the elements of the monodromy. The formula for the Hurwitz character χ is

$$\chi = 2 \cdot 1_G + \chi_{\langle 1_G \rangle} - \chi_{\langle g_2 \rangle} - \chi_{\langle g_3 \rangle} - \chi_{\langle g_7 \rangle} \tag{4}$$

where $\chi_{\langle h \rangle}$ for $h \in G$ is defined as follows:

$$\chi_{\langle h \rangle}(g) = \frac{1}{|\langle h \rangle|} \sum_{x \in G} \chi^o(xgx^{-1}), \text{ where } \chi^o(g) = \begin{cases} 1 \text{ if } g \in \langle h \rangle \\ 0 \text{ if } g \notin \langle h \rangle. \end{cases}$$

If we let $\chi' = \chi_{\langle 1_G \rangle} - \chi_{\langle g_2 \rangle} - \chi_{\langle g_3 \rangle} - \chi_{\langle g_7 \rangle}$ then $\langle \chi, \varphi_i \rangle = \langle \chi', \varphi_i \rangle$ if $\varphi_i \neq 1_G$ so we compute χ' instead of χ. Also note that by its definition, $\chi'(g) = 0$ if g is not of order 1, 2, 3, or 7.

Table 2 : Values of χ' on conjugacy classes of elements of order 1, 2, 3, and 7.

q	Value for elements of order				q	Value for elements of order			
mod84	1	2	3	7	mod84	1	2	3	7
1	$\frac{\lvert G\rvert}{42}$	$-\left(\frac{q-1}{2}\right)$	$-\left(\frac{q-1}{3}\right)$	$-\left(\frac{q-1}{7}\right)$	-1	$\frac{\lvert G\rvert}{42}$	$-\left(\frac{q+1}{2}\right)$	$-\left(\frac{q+1}{3}\right)$	$-\left(\frac{q+1}{7}\right)$
13	$\frac{\lvert G\rvert}{42}$	$-\left(\frac{q-1}{2}\right)$	$-\left(\frac{q-1}{3}\right)$	$-\left(\frac{q+1}{7}\right)$	-13	$\frac{\lvert G\rvert}{42}$	$-\left(\frac{q+1}{2}\right)$	$-\left(\frac{q+1}{3}\right)$	$-\left(\frac{q-1}{7}\right)$
29	$\frac{\lvert G\rvert}{42}$	$-\left(\frac{q-1}{2}\right)$	$-\left(\frac{q+1}{3}\right)$	$-\left(\frac{q-1}{7}\right)$	-29	$\frac{\lvert G\rvert}{42}$	$-\left(\frac{q+1}{2}\right)$	$-\left(\frac{q-1}{3}\right)$	$-\left(\frac{q+1}{7}\right)$
43	$\frac{\lvert G\rvert}{42}$	$-\left(\frac{q+1}{2}\right)$	$-\left(\frac{q-1}{3}\right)$	$-\left(\frac{q-1}{7}\right)$	-43	$\frac{\lvert G\rvert}{42}$	$-\left(\frac{q-1}{2}\right)$	$-\left(\frac{q+1}{3}\right)$	$-\left(\frac{q+1}{7}\right)$

Inner Product Computations

We now compute the inner product of the Hurwitz character with the irreducible $\mathbb{Q}$-characters of G. Since $\langle \chi, 1_G \rangle = 0$ and all other irreducible $\mathbb{Q}$-characters have degree greater than 1,

Proposition

No Hurwitz curve with automorphism group PSL$(2, q)$ has a simple Jacobian variety.

Now $\langle \chi, \lambda \rangle = \frac{q-u}{42}$ and $\langle \chi, \chi_1 + \chi_2 \rangle = \frac{q-v}{42}$ where u and v are given in Table 3. For every $d \mid \frac{q-1}{2}$ and $d < \frac{q-5}{4}$ if $q \equiv 1 \bmod 4$, or $d < \frac{q-3}{4}$ if $q \equiv -1 \bmod 4$, there is an irreducible $\mathbb{Q}$-character which is the sum of several μ_k.

Table 3 : Values of u and v for $\langle \chi, \lambda \rangle$ and $\langle \chi, \chi_1 + \chi_2 \rangle$, respectively.

$q \bmod 168$	u	v	$q \bmod 168$	u	v
± 1	± 85	± 169	$\pm 43 \bmod 168$	± 43	± 43
± 13	± 13	± 13	$\pm 85 \bmod 168$	± 85	± 85
± 29	± 29	± 29	$\pm 97 \bmod 168$	± 13	± 97
± 41	∓ 43	± 41	$\pm 113 \bmod 168$	± 29	± 113

The inner product of this character with χ is given by $\frac{\phi\left(\frac{q-1}{2d}\right)(q-w_\mu)}{84}$ where w_μ is determined by the least residue of $q \bmod 84$ as well as the number of 2, 3, and 7 which divide d, and ϕ is Euler's phi-function.

Similar results hold for $\langle \chi, \theta_t \rangle$ with some constant w_θ.

Results

We now combine (1) and the inner product computations from the previous section.

Theorem

Let X be a Hurwitz curve with full automorphism group PSL$(2, q)$ where q is odd, $q > 27$, and q satisfies one of (2) or (3) in the Theorem above. Let A_x^y denote the product of y copies of an abelian variety of dimension x. Let u, v, w_μ and w_θ be as described above.

When $q \equiv 1 \bmod 4$ then the Jacobian variety of X is isogenous to

$$A^q_{\frac{q-u}{84}} \times A^{\frac{q+1}{2}}_{\frac{q-v}{84}} \times \prod_{\substack{d \mid \frac{q-1}{2} \\ d < \frac{q-5}{4}}} A^{q+1}_{\frac{\phi\left(\frac{q-1}{2d}\right)\cdot(q-w_\mu)}{168}} \times \prod_{\substack{d \mid \frac{q+1}{2} \\ d < \frac{q-1}{4}}} A^{q-1}_{\frac{\phi\left(\frac{q+1}{2d}\right)\cdot(q-w_\theta)}{168}}.$$

When $q \equiv -1 \bmod 4$, then the Jacobian variety of X is isogenous to

$$A^q_{\frac{q-u}{84}} \times A^{\frac{q-1}{2}}_{\frac{q-v}{84}} \times \prod_{\substack{d \mid \frac{q-1}{2} \\ d < \frac{q-3}{4}}} A^{q+1}_{\frac{\phi\left(\frac{q-1}{2d}\right)\cdot(q-w_\mu)}{168}} \times \prod_{\substack{d \mid \frac{q+1}{2} \\ d < \frac{q-3}{4}}} A^{q-1}_{\frac{\phi\left(\frac{q+1}{2d}\right)\cdot(q-w_\theta)}{168}}.$$

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