Elliptic Factors in the Jacobian Varieties of Low Genus Curves

Jennifer Paulhus

Kansas State University
paulhus@math.ksu.edu
www.math.ksu.edu/$\sim$paulhus

January 8, 2008

X denotes a curve of genus g, a smooth projective variety of dimension 1. J_X its Jacobian variety, E an elliptic curve and A_d an abelian variety of dimension d.

ζ_n a primitive n-th root of unity

Question

Given a genus g, is there a curve X of that genus such that $J_X \sim E^g$?

Question

Given a genus g, what is the largest integer t such that there is a curve X of genus g with t copies of an elliptic curve E in the decomposition of J_X?

G is a group contained in the automorphism group of the curve X.

D_n, C_n are the dihedral and cyclic groups of order n, respectively.

Example

$X : y^2 = x(x^6 + x^3 + 1)$

$\mathit{Aut}(X) = D_{12} = \langle r, s \mid r^6, s^2, (rs)^2 \rangle$ *where*

$r : (x, y) \to (\zeta_3 x, \zeta_6 y) \qquad s : (x, y) \to (1/x, y/x^4)$

$\mathrm{End}_0(J_X) := \mathrm{End}(J_X) \otimes_{\mathbb{Z}} \mathbb{Q}$

Natural map of $\mathbb{Q}$-algebras $e : \mathbb{Q}[G] \to \mathrm{End}_0(J_X)$

Definition

Given $\varepsilon_1, \varepsilon_2 \in$ *End*$_0(J_X)$, $\varepsilon_1 \sim \varepsilon_2$ *if* $\chi(\varepsilon_1) = \chi(\varepsilon_2)$ *for all virtual characters* χ *in End*$_0 J_X$.

Theorem (Kani-Rosen, 1989)

If $\varepsilon_i, \varepsilon'_j \in$ *End*$_0(J_X)$, *then*

$$\varepsilon_1 + \cdots + \varepsilon_m \sim \varepsilon'_1 + \cdots + \varepsilon'_n$$

if and only if

$$\varepsilon_1(J_X) \times \cdots \times \varepsilon_m(J_X) \sim \varepsilon'_1(J_X) \times \cdots \times \varepsilon'_n(J_X).$$

$$
\begin{array}{ccc}
\mathbb{Q}[G] & & 1_{\mathbb{Q}[G]} \\
\downarrow & \{e\} & \downarrow \\
\mathsf{End}_0 J_X & & 1_{\mathsf{End}_0 J_X} \\
\downarrow & \{\text{Kani-Rosen}\} & \downarrow \\
\text{Jacobian isogenies} & & J_X
\end{array}
$$

In particular we want to find idempotent relations in $\mathbb{Q}[G]$ which contain the identity element.

From a theorem of Wedderburn we know that

$$\mathbb{Q}[G] \cong \bigoplus_i M_{n_i}(\Delta_i)$$

where Δ_i are division rings.

$\pi_{i,j}$ is the element of $\mathbb{Q}[G]$ with the zero matrix in every component except the ith component where it has a 1 in the j,j position and zeros elsewhere.

$$\mathbf{1}_{\mathbb{Q}[G]} = \sum_{i,j} \pi_{i,j}$$

$$J_X \sim \bigoplus_{i,j} e(\pi_{i,j}) J_X$$

Suppose the quotient map from X to $Y = X/G$ is branched at s points with monodromy $g_1, \ldots, g_s \in G$.

$\chi_{\langle g_i \rangle}$ is the character of G which is induced from the trivial character of $\langle g_i \rangle$ and χ_{triv} is the trivial character of G.

Definition

A **Hurwitz character** *of a group G is a character of the form:*

$$\chi = 2\chi_{\mathit{triv}} + 2\left(g_Y - 1\right)\chi_{\langle 1_G \rangle} + \sum_{i=1}^{s} \left(\chi_{\langle 1_G \rangle} - \chi_{\langle g_i \rangle}\right)$$

V is the representation associated to this character and the V_i (with associated character χ_i) are the irreducible $\mathbb{Q}$-representations.

$$\text{dim } e(\pi_{i,j})J_X = \tfrac{1}{2}\ \text{dim}_{\mathbb{Q}}\pi_{i,j}V$$

and

$$\text{dim}_{\mathbb{Q}}\pi_{i,j}V = \langle \chi_i, \chi \rangle$$

Recall: We want to find lots of isogenous elliptic curves.

Theorem (Paulhus, 2007)

With notation as above, $e(\pi_{i,j})J_X$ is isogenous to $e(\pi_{i,k})J_X$.

This theorem suggests we should find curves of genus g whose automorphism groups have an $M_g(\Delta_i)$ somewhere in the Wedderburn decomposition....

...or, if no such curve of a certain genus exists, at least try to maximize t in $M_t(\Delta_i)$.

Work of Magaard, Shaska, Shpectorov, and Völklein (2002) classifies all full automorphism groups of "large" curves up to genus 10.

Large in their paper means $|G| > 4(g-1)$. In particular X/G is genus 0 in these cases.

Data in their paper provides information about monodromy of the quotient maps as well as dimensions of the families of curves with each particular automorphism group.

In genus 5 there is one curve, up to isomorphism, with $\mathrm{Aut}(X) = G$, the GAP group $(160, 234)$. Monodromy of this cover consists of an order 2, 4, and 5 element.

$$\mathbb{Q}[G] \cong 2\mathbb{Q} \oplus M_2(\mathbb{Q}(\zeta_5 + \zeta_5^{-1})) \oplus 6M_5(\mathbb{Q})$$

$$J_X \sim e(\pi_{1,1})J_X \times e(\pi_{2,1})J_X \times e(\pi_{3,1})J_X \times e(\pi_{3,2})J_X \times e(\pi_{4,1})J_X \times \cdots$$

$$\times e(\pi_{4,5})J_X \times e(\pi_{5,1})J_X \times \cdots \times e(\pi_{9,1})J_X \times \cdots \times e(\pi_{9,5})J_X$$

$\langle \chi_1, \chi \rangle = \langle \chi_2, \chi \rangle = 0$ and dim $J_X = g = 5$ and so, by a simple dimension argument, the dimension of $e(\pi_{i,j})J_X$ for i equal to one of 4 through 9 must be one, while the dimension of the others, as well as $i = 3$ must be zero so $J_X \sim E^5$.

Genus	Auto. Group	Dim.	Jacobian Decomposition
4	$(72, 40)$	0	$J_X \sim E^4$
5	$(160, 234)$	0	$J_X \sim E^5$
6	$(72, 15)$	0	$J_X \sim E^6$
7	$(32, 43)$	1	$J_X \sim E_1 \times E_2^2 \times E_3^4$
	$S_3 \times S_3$	1	
	$S_3 \times D_8$	1	
8	$(32, 18)$	1	$J_X \sim E_1^2 \times E_2^2 \times A_4$
9	$(192, 955)$	0	$J_X \sim E_1^3 \times E_2^6$
10	$(72, 40)$	1	$J_X \sim E_1^2 \times E_2^4 \times E_3^4$

G the automorphism group of a hyperelliptic curve X, ω the hyperelliptic involution

The reduced automorphism group of a hyperelliptic curve ($G/\langle\omega\rangle$) must be one of D_n, C_n, A_4, S_4, A_5.

Work of Brandt and Stichtenoth (1986) and Shaska (2003) completely classifies all possible full automorphism groups of hyperelliptic curves over an algebraically closed field of characteristic zero for any genus.

Shaska (2003) lists affine models for the curves with any given genus and monodromy for the coverings.

For any genus g there is at most one family of hyperelliptic curves of that genus with reduced automorphism group each of A_4, S_4, or A_5. This existence is completely determined by the residue class of g modulo 6, 12, and 30, respectively.

For A_4, the full automorphism groups are $C_2 \times A_4$ or $SL_2(3)$.

For S_4, the full automorphism groups are $GL_2(3)$, $C_2 \times S_4$ or the following groups of order 48: $W_2 = \langle a, b \mid a^4, b^3, ba^2b^{-1}a^2, (ab)^4 \rangle$ and $W_3 = \langle a, b \mid a^4, a^2(ab)^4, (ab)^8 \rangle$.

For A_5, the full automorphism groups are $C_2 \times A_5$ or $SL_2(5)$.

Genus	Auto. Group	Dim.	Jacobian Decomposition
3	$S_4 \times C_2$	0	E^3
4	$SL_2(3)$	0	$E_1^2 \times E_2^2$
5	$A_4 \times C_2$	1	$A_2 \times E^3$
	W_2	0	$E_1^2 \times E_2^3$
	$A_5 \times C_2$	0	E^5
6	$GL_2(3)$	0	$E_1^2 \times E_2^4$
7	$A_4 \times C_2$	1	$E \times A_2^3$
8	$SL_2(3)$	1	$A_{2,1}^2 \times A_{2,2}^2$
	W_3	0	$A_2^2 \times E^4$
9	$A_4 \times C_2$	1	$A_2^3 \times E^3$
	W_2	0	$E_1 \times E_2^2 \times A_2^3$
	$A_5 \times C_2$	0	$E_1^4 \times E_2^5$
10	$SL_2(3)$	1	$A_2^2 \times A_3^2$

Theorem (Paulhus)

The genus 5 hyperelliptic curve with affine model

$$X : y^2 = x(x^{10} + 11x^5 - 1)$$

has automorphism group $A_5 \times C_2$ and $J_X \sim E^5$ for some elliptic curve E.

Proof.

The irreducible $\mathbb{Q}$-characters of this group consist of 2 each of degree 1, 3, 4, and 5 characters.

Computing the Hurwitz character and then the inner product of each of the irreducible $\mathbb{Q}$-characters with it results in a value of zero for all except one of the degree 5 characters where the inner product is a 2.

$\square$

The End